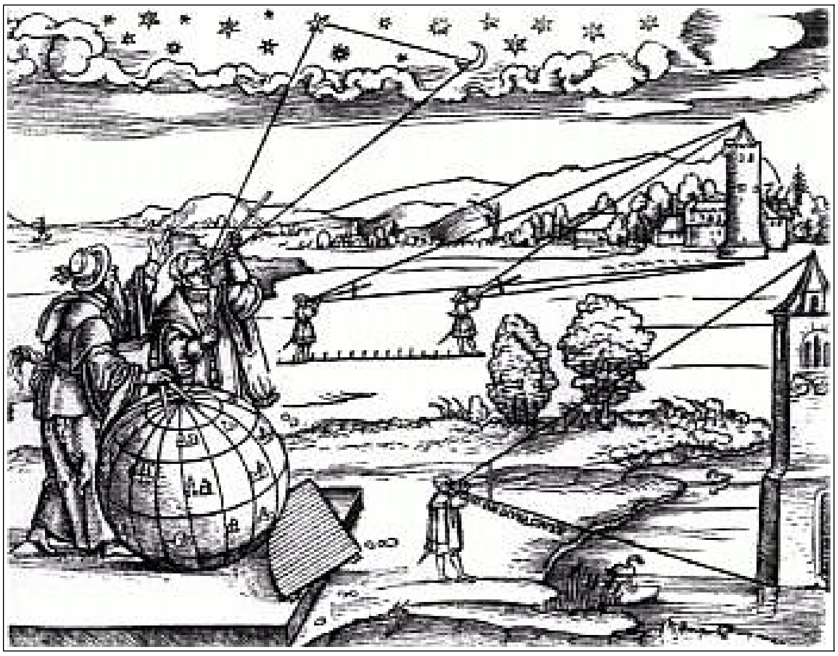
A Short Guide to

Celestial Navigation



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May 24, 2001

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Legal Notice

Carpe diem.
Horace

Preface

Why should anybody still use celestial navigation in the era of electronics and GPS? You might as well ask why some people still develop black and white photos in their darkroom instead of using a high-color digital camera and image processing software. The answer would be the same: because it is a noble art, and because it is fun. Reading a GPS display is easy and not very exciting as soon as you have got used to it. Celestial navigation, however, will always be a challenge because each scenario is different. Finding your geographic position by means of astronomical observations requires knowledge, judgement, and the ability to handle delicate instruments. In other words, you play an active part during the whole process, and you have to use your brains. Everyone who ever reduced a sight knows the thrill I am talking about. The way is the goal.
It took centuries and generations of navigators, astronomers, geographers, mathematicians, and instrument makers to develop the art and science of celestial navigation to its present state, and the knowledge thus accumulated is too precious to be forgotten. After all, celestial navigation will always be a valuable alternative if a GPS receiver happens to fail.
Years ago, when I read my first book on navigation, the chapter on celestial navigation with its fascinating diagrams and formulas immediately caught my particular interest although I was a little deterred by its complexity at first. As I became more advanced, I realized that celestial navigation is not as difficult as it seems to be at first glance. Further, I found that many publications on this subject, although packed with information, are more confusing than enlightening, probably because most of them have been written by experts and for experts.
I decided to write something like a compact guide-book for my personal use which had to include operating instructions as well as all important formulas and diagrams. The idea to publish it came in 1997 when I became interested in the internet and found that it is the ideal medium to share one's knowledge with others. I took my manuscript, rewrote it in the form of a structured manual, and redesigned the layout to make it more attractive to the public. After converting everything to the HTML format, I published it on my web site. Since then, I have revised text and graphic images several times and added a couple of new chapters. People seem to like it, at least I get approving e-mails now and then.

Following the recent trend, I decided to convert the manual to the PDF format, which has become an established standard for internet publishing. In contrast to HTML documents, the page-oriented PDF documents retain their layout when printed. The HTML version is no longer available since keeping two versions in different formats synchronized was too much work. In my opinion, a printed manual is more useful anyway.

Since people keep asking me how I wrote the documents and how I created the graphic images, a short description of the procedure and software used is given below:

Drawings and diagrams were made with good old CorelDraw! 3.0 and exported as GIF files.

The manual was designed and written with Star Office. The Star Office (.sdw) documents were then converted to Postscript (.ps) files with the AdobePS printer driver (available at www.adobe.com). Finally, the Postscript files were converted to PDF files with GsView and Ghostscript (www.ghostscript.com).

I apologize for misspellings, grammar errors, and wrong punctuation. I did my best, but after all, English is not my native language.
I hope the new version will find as many readers as the old one. Hints and suggestions are always welcome. Since I am very busy, I may not always be able to answer incoming e-mails immediately. Be patient.
Last but not least, I owe my wife an apology for spending countless hours in front of the PC, staying up late, neglecting household chores, etc. I'll try to mend my ways. Some day ...

May 24, 2001

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Chapter 1

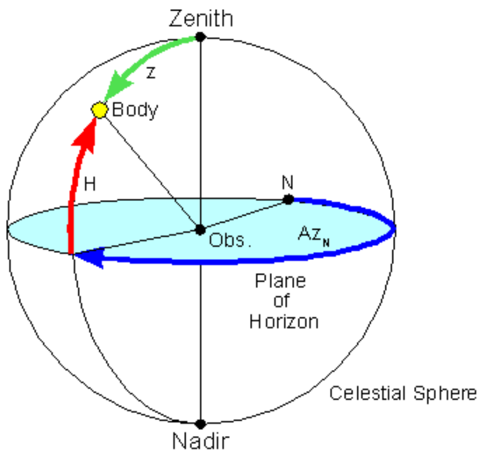
The Elements of Celestial Navigation

Celestial navigation, a branch of applied astronomy, is the art and science of finding one's geographic position through astronomical observations, particularly by measuring **altitudes** of celestial bodies – sun, moon, planets, or stars.

An observer watching the night sky without knowing anything about geography and astronomy might spontaneously get the impression of being on a plane located at the center of a huge, hollow sphere with the celestial bodies attached to its inner surface. Indeed, this naive model of the universe was in use for millennia and developed to a high degree of perfection by ancient astronomers. Still today, it is a useful tool for celestial navigation since the navigator, like the astronomers of old, measures **apparent positions** of bodies in the sky but not their absolute positions in space.

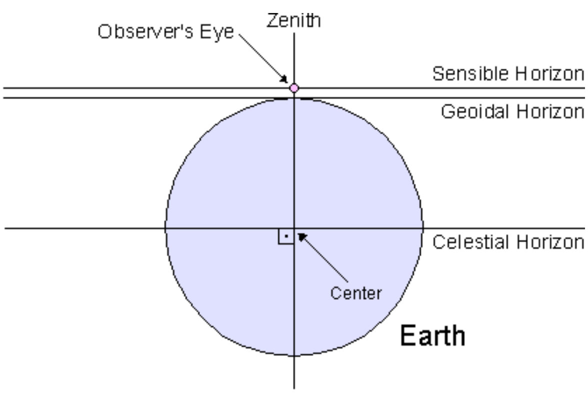
Following the above scenario, the apparent position of a body in the sky is defined by the **horizon system of coordinates**. In this system, the observer is located at the center of a fictitious hollow sphere of infinite diameter, the **celestial sphere**, which is divided into two hemispheres by the plane of the **celestial horizon** (*Fig. 1-1*). The **altitude**, **H**, is the vertical angle between the line of sight to the respective body and the celestial horizon, measured from 0° through +90° when the body is above the horizon (visible) and from 0° through -90° when the body is below the horizon (invisible). The **zenith distance**, **z**, is the corresponding angular distance between the body and the **zenith**, an imaginary point vertically overhead. The zenith distance is measured from 0° through 180°. The point opposite to the zenith is called **nadir** (z = 180°). H and z are **complementary angles** (H + z = 90°). The **azimuth**, **Az_N**, is the horizontal direction of the body with respect to the geographic (true) north point on the horizon, measured clockwise from 0° through 360°.

Fig. 1-1



In reality, the observer is not located at the celestial horizon but at the the **sensible horizon.** *Fig. 1-2* shows the three horizontal planes relevant to celestial navigation:

Fig. 1-2



The **sensible horizon** is the horizontal plane passing through the observer's eye. The **celestial horizon** is the horizontal plane passing through the center of the earth which coincides with the center of the celestial sphere. Moreover, there is the **geoidal horizon,** the horizontal plane tangent to the earth at the observer's position. These three planes are parallel to each other.

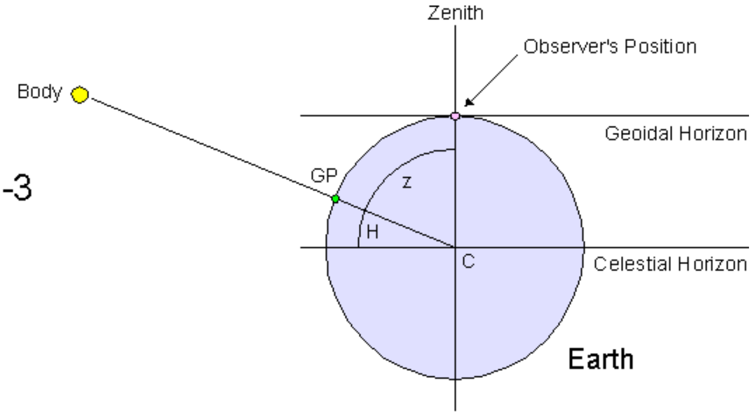
The sensible horizon merges into the geoidal horizon when the observer's eye is at sea or ground level. Since both horizons are usually very close to each other, they can be considered as identical under practical conditions. None of the above horizontal planes coincides with the **visible horizon**, the line where the earth's surface and the sky appear to meet.

Calculations of celestial navigation always refer to the geocentric altitude of a body, the altitude with respect to a fictitious observer being at the celestial horizon and at the center of the earth which coincides the center of the celestial sphere. Since there is no way to measure this altitude directly, it has to be derived from the altitude with respect to the visible or sensible horizon (altitude corrections, chapter 2).

A **marine sextant** is an instrument designed to measure the altitude of a body with reference to the visible sea horizon. Instruments with any kind of an **artificial horizon** measure the altitude referring to the sensible horizon (chapter 2).

Altitude and zenith distance of a celestial body depend on the distance between a terrestrial observer and the **geographic position of the body**, **GP**. GP is the point where a straight line from the body to the center of the earth, C, intersects the earth's surface (*Fig. 1-3*).

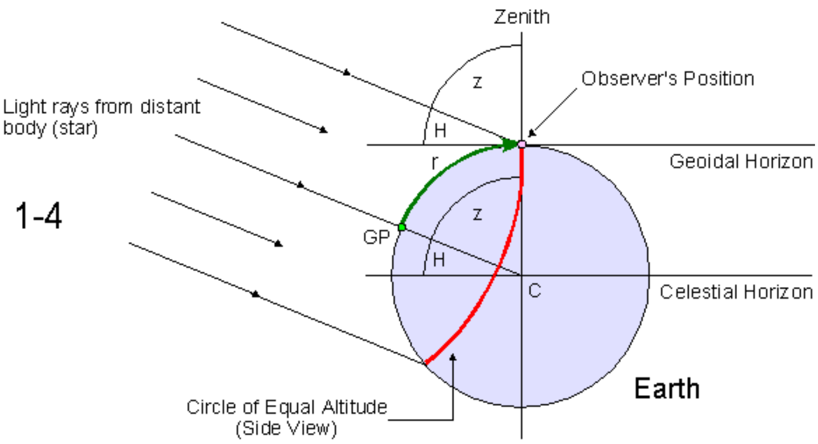
Fig. 1-3



A body appears in the zenith (z = 0°, H = 90°) when GP is identical with the observer's position. A terrestrial observer moving away from GP will observe that the altitude of the body decreases as his distance from GP increases. The body is on the celestial horizon (H = 0°, z = 90°) when the observer is one quarter of the circumference of the earth away from GP.

For a given altitude of a body, there is an infinite number of positions having the same distance from GP and forming a circle on the earth's surface whose center is on the line C–GP, below the earth's surface. Such a circle is called a **circle of equal altitude**. An observer traveling along a circle of equal altitude will measure a constant altitude and zenith distance of the respective body, no matter where on the circle he is. The radius of the circle, r, measured along the surface of the earth, is directly proportional to the observed zenith distance, z (*Fig 1-4*).

Fig. 1-4



$$r\left[nm\right] = 60 \cdot z\left[°\right] \qquad or \qquad r\left[km\right] = \frac{Perimeter\ of\ Earth\ \left[km\right]}{360°} \cdot z\left[°\right]$$

One nautical mile (1 nm = 1.852 km) is the **great circle distance** of one minute of arc (the definition of a great circle is given in chapter 3). The mean perimeter of the earth is 40031.6 km.

Light rays coming from distant objects (stars) are virtually parallel to each other when reaching the earth. Therefore, the altitude with respect to the geoidal (sensible) horizon equals the altitude with respect to the celestial horizon. In contrast, light rays coming from the relatively close bodies of the solar system are diverging. This results in a measurable difference between both altitudes (parallax). The effect is greatest when observing the moon, the body closest to the earth (see chapter 2, *Fig. 2-4*).

The azimuth of a body depends on the observer's position on the circle of equal altitude and can assume any value between 0° and 360°.

Whenever we measure the altitude or zenith distance of a celestial body, we have already gained partial information about our own geographic position because we know we are somewhere on a circle of equal altitude with the radius r and the center GP, the geographic position of the body. Of course, the information available so far is still incomplete because we could be anywhere on the circle of equal altitude which comprises an infinite number of possible positions and is therefore also called a **circle of position** (see chapter 4).

We continue our mental experiment and observe a second body in addition to the first one. Logically, we are on two circles of equal altitude now. Both circles overlap, intersecting each other at two points on the earth's surface, and one of those two points of intersection is our own position (*Fig. 1-5a*). Theoretically, both circles could be tangent to each other, but this case is highly improbable (see chapter 16).

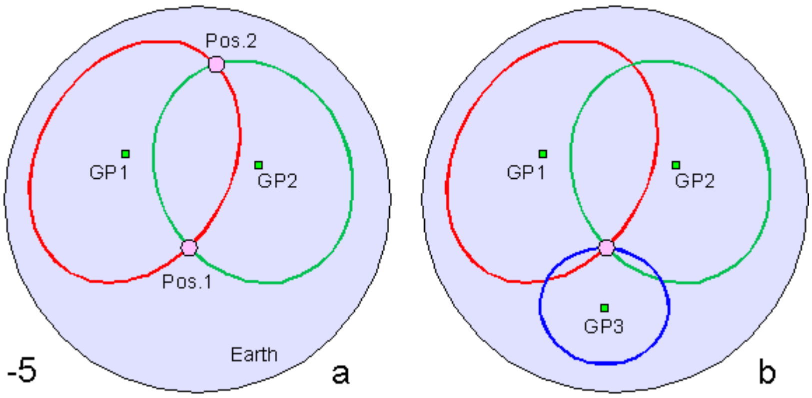


Fig. 1-5

In principle, it is not possible to know which point of intersection – Pos.1 or Pos.2 – is identical with our actual position unless we have additional information, e.g., a fair estimate of where we are, or the **compass bearing** of at least one of the bodies. Solving the problem of **ambiguity** can also be achieved by observation of a third body because there is only one point where all three circles of equal altitude intersect (*Fig. 1-5b*).

Theoretically, we could find our position by plotting the circles of equal altitude on a globe. Indeed, this method has been used in the past but turned out to be impractical because precise measurements require a very big globe. Plotting circles of equal altitude on a map is possible if their radii are small enough. This usually requires observed altitudes of almost 90°. The method is rarely used since such altitudes are not easy to measure. In most cases, circles of equal altitude have diameters of several thousand nautical miles and can **not** be plotted on usual maps. Further, plotting circles on a map is made more difficult by geometric distortions related to the map projection (chapter 13).

Since a navigator always has an estimate of his position, it is not necessary to plot the whole circles of equal altitude but rather their parts near the expected position. In the 19th century, two ingenious navigators developed ways to construct straight lines (secants and tangents of the circles of equal altitude) whose point of intersection approximates our position. These revolutionary methods, which marked the beginning of modern celestial navigation, will be explained later. In summary, finding one's position by astronomical observations includes three basic steps:

1. Measuring the altitudes or zenith distances of two or more chosen bodies (chapter 2).

2. Finding the geographic position of each body at the time of its observation (chapter 3).

3. Deriving the position from the above data (chapter 4&5).

Chapter 2

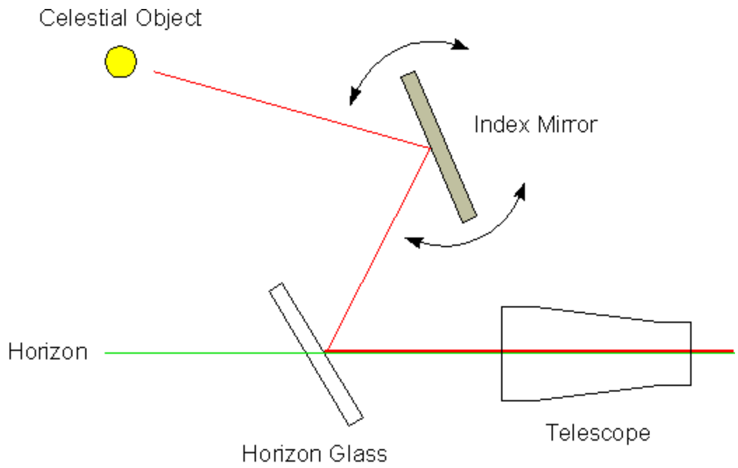
Altitude Measurement

Although altitudes and zenith distances are equally suitable for navigational calculations, most formulas are traditionally based upon altitudes which are easily accessible using the visible sea horizon as a natural reference line. Direct measurement of the zenith distance, however, requires an instrument with an artificial horizon, e.g., a pendulum or spirit level indicating the direction of the normal force (perpendicular to the local horizontal plane), since a reference point in the sky does not exist.

Instruments

A **marine sextant** consists of a system of two mirrors and a telescope mounted on a metal frame. A schematic illustration (side view) is given in *Fig. 2-1*. The rigid horizon glass is a semi-translucent mirror attached to the frame. The fully reflecting index mirror is mounted on the so-called index arm rotatable on a pivot perpendicular to the frame. When measuring an altitude, the instrument frame is held in a vertical position, and the visible sea horizon is viewed through the scope and horizon glass. A light ray coming from the observed body is first reflected by the index mirror and then by the back surface of the horizon glass before entering the telescope. By slowly rotating the index mirror on the pivot the superimposed image of the body is aligned with the image of the horizon. The corresponding altitude, which is twice the angle formed by the planes of horizon glass and index mirror, can be read from the graduated limb, the lower, arc-shaped part of the sextant frame (not shown). Detailed information on design, usage, and maintenance of sextants is given in [3] (see appendix).

Fig. 2-1



On land, where the horizon is too irregular to be used as a reference line, altitudes have to be measured by means of instruments with an artificial horizon:

A **bubble attachment** is a special sextant telescope containing an **internal artificial horizon** in the form of a small spirit level whose image, replacing the visible horizon, is superimposed with the image of the body. Bubble attachments are expensive (almost the price of a sextant) and not very accurate because they require the sextant to be held absolutely still during an observation, which is difficult to manage. A sextant equipped with a bubble attachment is referred to as a **bubble sextant**. Special bubble sextants were used for air navigation before electronic navigation systems became standard equipment.

A pan filled with water, or preferably an oily liquid like glycerol, can be utilized as an **external artificial horizon**. Due to the gravitational force, the surface of the liquid forms an exactly horizontal mirror unless distorted by vibrations or wind. The vertical angular distance between a body and its mirror image, measured with a marine sextant, is twice the altitude. This very accurate method is the perfect choice for exercising celestial navigation in a backyard.

A **theodolite** is basically a telescopic sight which can be rotated about a vertical and a horizontal axis. The angle of elevation is read from the vertical circle, the horizontal direction from the horizontal circle. Built-in spirit levels are used to align the instrument with the plane of the sensible horizon before starting the observations (artificial horizon). Theodolites are primarily used for surveying, but they are excellent navigation instruments as well. Many models can measure angles to 0.1' which cannot be achieved even with the best sextants. A theodolite is mounted on a tripod and has to stand on solid ground. Therefore, it is restricted to land navigation. Traditionally, theodolites measure zenith distances. Modern models can optionally measure altitudes.

Never view the sun through an optical instrument without inserting a proper shade glass, otherwise your eye might suffer permanent damage !

Altitude corrections

Any altitude measured with a sextant or theodolite contains errors. Altitude corrections are necessary to eliminate systematic altitude errors and to reduce the altitude measured relative to the visible or sensible horizon to the altitude with respect to the celestial horizon and the center of the earth (chapter 1). Of course, altitude corrections do not remove random errors.

Index error (IE)

A sextant or theodolite, unless recently calibrated, usually has a constant error (**index error, IE**) which has to be subtracted from the readings before they can be processed further. The error is positive if the displayed value is greater than the actual value and negative if the displayed value is smaller. Angle-dependent errors require alignment of the instrument or the use of an individual correction table.

$$1st\ correction: \qquad H_1 = Hs - IE$$

The **sextant altitude**, **Hs**, is the altitude as indicated by the sextant before any corrections have been applied.

When using an external artificial horizon, H_1 (not Hs!) has to be divided by two.

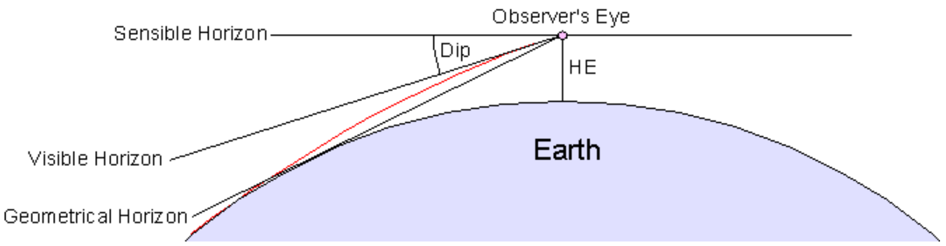
A theodolite measuring the zenith distance, z, requires the following formula to obtain H_1:

$$H_1 = 90° - (z - IE)$$

Dip of horizon

If the earth's surface were an infinite plane, visible and sensible horizon would be identical. In reality, the visible horizon appears several arcminutes below the sensible horizon which is the result of two contrary effects, the curvature of the earth's surface and atmospheric refraction. The **geometrical horizon**, a flat cone, is formed by an infinite number of straight lines tangent to the earth and radiating from the observer's eye. Since atmospheric refraction bends light rays passing along the earth's surface toward the earth, all points on the geometric horizon appear to be elevated, and thus form the visible horizon. If the earth had no atmosphere, the visible horizon would coincide with the geometrical horizon (*Fig. 2-2*).

Fig. 2-2



The altitude of the sensible horizon relative to the visible horizon is called **dip** and is a function of the **height of eye**, **HE**, the vertical distance of the observer's eye from the earth's surface:

$$Dip\,[']\approx 1.76\cdot\sqrt{HE\,[m]}\approx 0.97\cdot\sqrt{HE\,[ft]}$$

The above formula is empirical and includes the effects of the curvature of the earth's surface and atmospheric refraction*.

*At sea, the dip of horizon can be obtained directly by measuring the vertical angle between the visible horizon in front of the observer and the visible horizon behind the observer (through the zenith). Subtracting 180° from the angle thus measured and dividing the resulting angle by two yields the dip of horizon. This very accurate method is rarely used because it requires a special instrument (similar to a sextant).

$$2nd\ correction: \qquad H_2 = H_1 - Dip$$

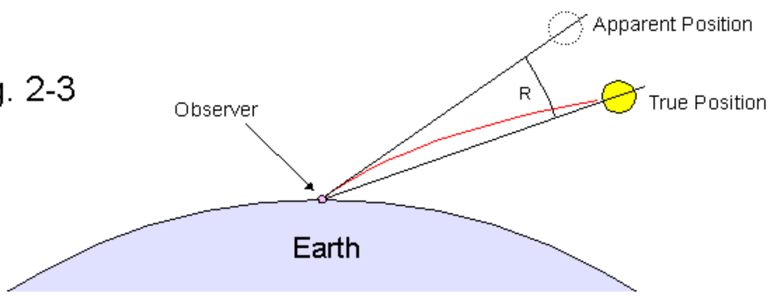
The correction for dip has to be omitted (dip = 0) if any kind of an artificial horizon is used since an artificial horizon indicates the sensible horizon.

The altitude obtained after applying corrections for index error and dip is also referred to as **apparent altitude**, **Ha**.

Atmospheric refraction

A light ray coming from a celestial body is slightly deflected toward the earth when passing obliquely through the atmosphere. This phenomenon is called **refraction**, and occurs always when light enters matter of different density at an angle smaller than 90°. Since the eye can not detect the curvature of the light ray, the body appears to be at the end of a straight line tangent to the light ray at the observer's eye and thus appears to be higher in the sky. R is the angular distance between apparent and true position of the body at the observer's eye (*Fig. 2-3)*.

Fig. 2-3



Refraction is a function of Ha (= H_2). Atmospheric **standard refraction**, $\mathbf{R_0}$, is 0' at 90° altitude and increases progressively to approx. 34' as the apparent altitude approaches 0°:

Ha [°]	0	1	2	5	10	20	30	40	50	60	70	80	90
R_0 [']	~34	~24	~18	9.9	5.3	2.6	1.7	1.2	0.8	0.6	0.4	0.2	0.0

R_0 can be calculated with a number of formulas like, e. g., *Smart's* formula which gives highly accurate results from 15° through 90° altitude [2, 9]:

$$R_0\,[']=0.97127\cdot\tan\left(90°-H_2[°]\right)-0.00137\cdot\tan^3\left(90°-H_2[°]\right)$$

For navigation, *Smart's* formula is still accurate enough at 10° altitude. Below 5°, the error increases progressively.

For altitudes between 0° and 15°, the following formula is recommended [10]. H_2 is measured in degrees:

$$R_0['] = \frac{34.133 + 4.197 \cdot H_2 + 0.00428 \cdot H_2^2}{1 + 0.505 \cdot H_2 + 0.0845 \cdot H_2^2}$$

A low-precision refraction formula including the whole range of altitudes from 0° through 90° was developed by *Bennett*:

$$R_0['] = \frac{1}{\tan\left(H_2[°] + \dfrac{7.31}{H_2[°] + 4.4} \right)}$$

The accuracy is sufficient for navigational purposes. The maximum systematic error, occurring at 12° altitude, is approx. 0.07' [2]. If necessary, *Bennett's* formula can be improved (max. error: 0.015') by the following correction:

$$R_{0,\ improved}['] = R_0['] - 0.06 \cdot \sin\left(14.7 \cdot R_0['] + 13\right)$$

The argument of the sine is stated in degrees [2].

Refraction is influenced by atmospheric pressure and air temperature. The standard refraction, R_0, has to be multiplied with a correction factor, f, to obtain the refraction for a given combination of pressure and temperature if high precision is required.

$$f = \frac{p[mbar]}{1010} \cdot \frac{283}{273 + T[°C]} = \frac{p[in.Hg]}{29.83} \cdot \frac{510}{460 + T[°F]}$$

P is the atmospheric pressure and T the air temperature. **Standard conditions** (f = 1) are **1010 mbar** (29.83 in) and **10°C** (50°F). The effects of air humidity are comparatively small and can be ignored.

Refraction formulas refer to a fictitious standard atmosphere with the most probable density gradient. The actual refraction may differ from the calculated one if abnormal atmospheric conditions are present (temperature inversion, mirage effects, etc.). Particularly at low altitudes, anomalies of the atmosphere gain influence. Therefore, refraction at altitudes below ca. 5° may become erratic, and calculated values are not always reliable. It should be mentioned that dip, too, is influenced by atmospheric refraction and may become unpredictable under certain meteorological conditions.

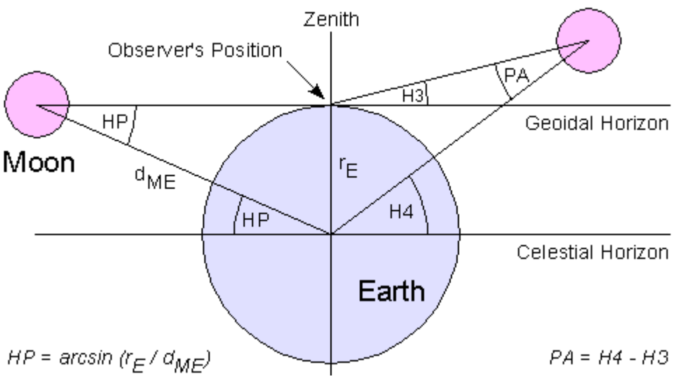
$$3rd\ correction: \quad H_3 = H_2 - f \cdot R_0$$

H_3 is the altitude of the body with respect to the sensible horizon.

Parallax

Calculations of celestial navigation refer to the altitude with respect to the earth's center and the celestial horizon. *Fig. 2-4* illustrates that the altitude of a near object, e.g., the moon, with respect to the celestial horizon, H_4, is noticeably greater than the altitude with respect to the geoidal (sensible) horizon, H_3. The difference H_4-H_3 is called **parallax in altitude**, **PA**. It decreases with growing distance between object and earth and is too small to be measured when observing stars (compare with chapter 1, *Fig. 1-4*). Theoretically, the observed parallax refers to the sensible, not to the geoidal horizon. Since the height of eye is several magnitudes smaller than the radius of the earth, the resulting error in parallax is not significant (< 0.0003' for the moon at 30 m height of eye).

Fig. 2-4



The parallax (in altitude) of a body being on the geoidal horizon is called **horizontal parallax**, **HP**. The HP of the sun is approx. 0.15'. Current HP's of the moon (ca. 1°!) and the navigational planets are given in the **Nautical Almanac** [12] and similar publications, e.g., [13]. PA is a function of altitude and HP of a body:

$$PA = \arcsin\left(\sin HP \cdot \cos H_3\right) \approx HP \cdot \cos H_3$$

When we observe the upper or lower limb of a body (see below), we assume that the parallax of the limb equals the parallax of the center (when at the same altitude). For geometric reasons (curvature of the surface), this is not quite correct. However, even with the moon, the body with by far the greatest parallax, the resulting error is so small that it can be ignored (<< 1").

The above formula is rigorous for a spherical earth. However, the earth is not exactly a sphere but resembles an **oblate spheroid**, a sphere flattened at the poles (chapter 9). This may cause a small but measurable error in the parallax of the moon (≤ 0.2'), depending on the observer's position [12]. Therefore, a small correction, OB, should be added to PA if high precision is required:

$$OB = \frac{HP}{298} \cdot \left[\sin\left(2 \cdot Lat\right) \cdot \cos Az_N \cdot \sin H_3 - \sin^2 Lat \cdot \cos H_3\right]$$

$$PA_{improved} = PA + OB$$

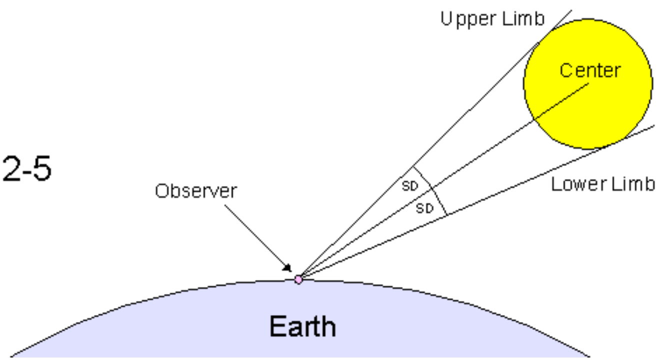
Lat is the observer's assumed latitude (chapter 4). $\mathbf{Az_N}$, the azimuth of the moon, is either measured with a compass (compass bearing) or calculated using the formulas given in chapter 4.

$$4th\ correction: \qquad H_4 = H_3 + PA$$

Semidiameter

When observing sun or moon with a marine sextant or theodolite, it is not possible to locate the center of the body with sufficient accuracy. It is therefore common practice to measure the altitude of the upper or lower limb of the body and add or subtract the apparent **semidiameter**, **SD**, the angular distance of the respective limb from the center (*Fig. 2-5*).

Fig. 2-5



We correct for the **geocentric** SD, the SD measured by a fictitious observer at the center the earth, since H_4 refers to the celestial horizon and the center of the earth (see *Fig. 2-4*). The geocentric semidiameters of sun and moon are given on the daily pages of the **Nautical Almanac** [12]. We can also calculate the geocentric SD of the moon from the tabulated horizontal parallax:

$$SD_{geocentric} = \arcsin\left(k \cdot \sin HP\right) \approx k \cdot HP \qquad k_{Moon} = 0.2725$$

The factor k is the ratio of the radius of the moon (1738 km) to the equatorial radius of the earth (6378 km).

Although the semidiameters of the navigational planets are not quite negligible (the SD of Venus can increase to 0.5'), the centers of these bodies are customarily observed, and no correction for SD is applied. Semidiameters of stars are much too small to be measured (SD=0).

$$5th\ correction: \qquad H_5 = H_4 \pm SD_{geocentric}$$

(lower limb: +SD, upper limb: –SD)

When using a bubble sextant which is less accurate anyway, we observe the center of the body and skip the correction for semidiameter.

The altitude obtained after applying the above corrections is called **observed altitude**, **Ho**.

$$Ho = H_5$$

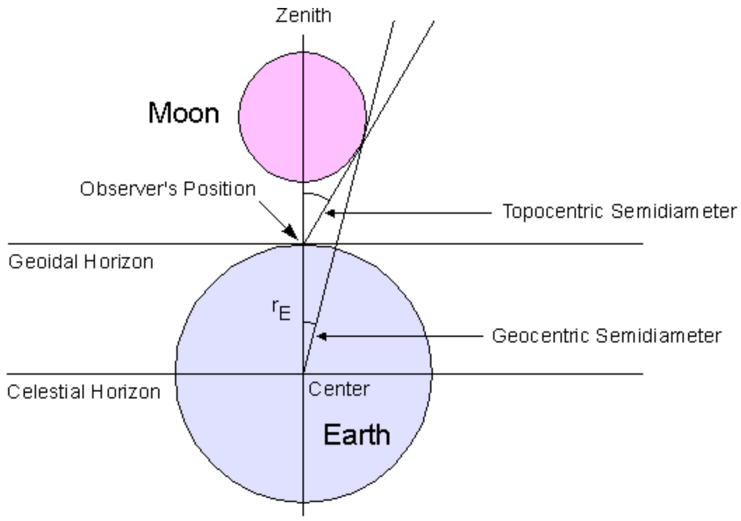
Ho is the geocentric altitude of the body, the altitude with respect to the celestial horizon and the center of the earth (see chapter 1).

Alternative corrections for semidiameter and parallax

The order of altitude corrections described above is in accordance with the Nautical Almanac. Alternatively, we can correct for semidiameter **before** correcting for parallax. In this case, however, we have to calculate with the **topocentric** semidiameter, the semidiameter of the respective body as seen from the observer's position on the surface of the earth (see *Fig. 2-5*), instead of the geocentric semidiameter.

With the exception of the moon, the body nearest to the earth, there is no significant difference between topocentric and geocentric SD. The topocentric SD of the moon is only marginally greater than the geocentric SD when the moon is on the sensible horizon but increases measurably as the altitude increases because of the decreasing distance between observer and moon. The distance is smallest (decreased by about the radius of the earth) when the moon is in the zenith. As a result, the topocentric SD of the moon being in the zenith is approximately 0.3' greater than the geocentric SD. This phenomenon is called **augmentation** (Fig. 2-6).

Fig. 2-6



The accurate formula for the topocentric (augmented) semidiameter of the moon is stated as:

$$SD_{topocentric} = \arctan \frac{k}{\sqrt{\frac{1}{\sin^2 HP} - \left(\cos H_3 \pm k\right)^2} - \sin H_3}$$

(lower limb: +k, upper limb: –k)

The following, simpler formula is accurate enough for navigational purposes (error << 1"):

$$SD_{topocentric} \approx k \cdot HP \cdot \frac{\cos H_3}{\cos\left(H_3 + HP \cdot \cos H_3\right)}$$

Thus, the fourth correction is:

$$4th\ correction\ (alt.): \qquad H_{4,\,alt} = H_3 \pm SD_{topocentric}$$

(lower limb: +SD, upper limb: –SD)

$H_{4,alt}$ is the topocentric altitude of the center of the moon.

Using one of the parallax formulas explained earlier, we calculate PA_{alt} from $H_{4,alt}$, and the fifth correction is:

$$5th\ correction\ (alt.): \qquad H_{5,\,alt} = H_{4,\,alt} + PA_{alt}$$

$$Ho = H_{5,\,alt}$$

Since the geocentric SD is easier to calculate than the topocentric SD, it is generally recommendable to correct for the semidiameter in the last place unless one has to know the augmented SD of the moon for special reasons.

Combined corrections for semidiameter and parallax of the moon

For observations of the moon, there is a surprisingly simple formula including the corrections for augmented semidiameter **as well as** parallax in altitude:

$$Ho = H_3 + \arcsin\left[\sin HP \cdot \left(\cos H_3 \pm k\right)\right]$$

(lower limb: +k, upper limb: –k)

The formula is rigorous for a spherical earth but does not take into account the effects of the flattening. Therefore, the small correction OB should be added to Ho.

To complete the picture, it should be mentioned that there is also a formula to calculate the topocentric (augmented) semidiameter of the moon from the geocentric altitude of the center, Ho:

$$SD_{topocentric} = \arcsin \frac{k}{\sqrt{1 + \frac{1}{\sin^2 HP} - 2 \cdot \frac{\sin Ho}{\sin HP}}}$$

Phase correction (Venus and Mars)

Since Venus and Mars show phases similar to the moon, their apparent center may differ somewhat from the actual center. Since the coordinates of both planets tabulated in the **Nautical Almanac** [12] refer to the apparent center, an additional correction is not required. The phase correction for Jupiter and Saturn is too small to be significant.

In contrast, coordinates calculated with **Interactive Computer Ephemeris** refer to the actual center. In this case, the upper or lower limb of the respective planet should be observed if the magnification of the telescope permits it.

The **Nautical Almanac** provides sextant altitude correction tables for sun, planets, stars (pages A2 – A4), and the moon (pages xxxiv – xxxv), which can be used instead of the above formulas if very high precision is not required (the tables cause additional rounding errors).
Instruments with an artificial horizon can exhibit additional errors caused by acceleration forces acting on the bubble or pendulum and preventing it from aligning itself with the direction of the gravitational force. Such acceleration forces can be random (vessel movements) or systematic (coriolis force). The coriolis force is important to air navigation and requires a special correction formula. In the vicinity of mountains, ore deposits, and other local irregularities of the earth's crust, the gravitational force itself can be slightly deflected from its normal direction.

Chapter 3

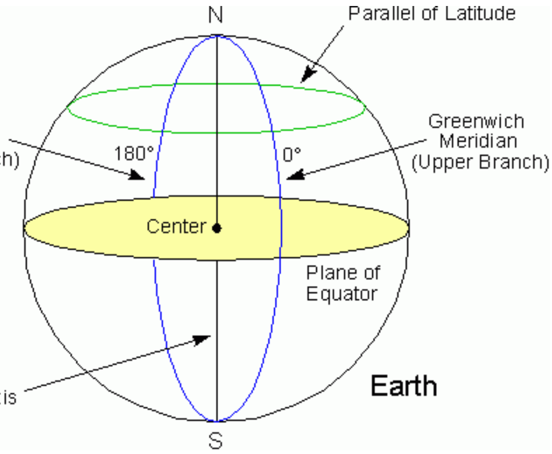
The Geographic Position (GP) of a Celestial Body

Geographic terms

In celestial navigation, the earth is regarded as a sphere. Although this is only an approximation, the geometry of the sphere is applied successfully, and the errors caused by the oblateness of the earth are usually negligible (see chapter 9).

Any circle on the surface of the earth whose plane passes through the center of the earth is called a **great circle**. Thus, a great circle is a circle with the greatest possible diameter on the surface of the earth. Any circle on the surface of the earth whose plane does not pass through the earth's center is called a **small circle**. The **equator** is the (only) great circle whose plane is perpendicular to the **polar axis**, the axis of rotation. Further, the equator is the only **parallel of latitude** being a great circle. Any other parallel of latitude is a small circle whose plane is parallel to the plane of the equator.
A **meridian** is a great circle going through the **geographic poles**, the points where the polar axis intersects the earth's surface. The **upper branch** of a meridian is the half from pole to pole passing through a given point, the **lower branch** is the opposite half. The **Greenwich meridian**, the meridian passing through the center of the transit instrument at the **Royal Greenwich Observatory**, was adopted as the **prime meridian** at the International Meridian Conference in October 1884. Its upper branch (0°) is the reference for measuring longitudes, its lower branch (180°) is known as the **International Dateline** (*Fig. 3-1*).

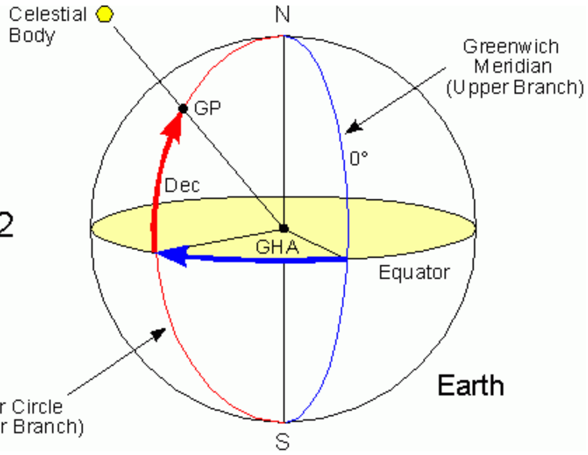
Fig. 3-1



Angles defining the position of a celestial body

The geographic position of a celestial body, GP, is defined by the **equatorial system of coordinates** (*Fig. 3-2*). The **Greenwich hour angle**, **GHA**, is the angular distance of GP **westward** from the upper branch of the Greenwich meridian (0°), measured from 0° through 360°. The **declination**, **Dec**, is the angular distance of GP from the plane of the equator, measured northward through +90° or southward through -90°. GHA and Dec are **geocentric coordinates** (measured at the center of the earth). The great circle going through the poles and GP is called **hour circle** (*Fig. 3-2*).

Fig. 3-2



GHA and Dec are equivalent to geocentric longitude and latitude with the exception that the longitude is measured from -(W)180° through +(E)180°.

Since the Greenwich meridian rotates with the earth from west to east, whereas each hour circle remains linked with the almost stationary position of the respective body in the sky, the GHA's of all celestial bodies increase as time progresses (approx. 15° per hour). In contrast to stars, the GHA's of sun, moon, and planets increase at slightly different (and variable) rates. This is attributable to the revolution of the planets (including the earth) around the sun and to the revolution of the moon around the earth, resulting in additional apparent motions of these bodies in the sky.

It is sometimes useful to measure the angular distance between the hour circle of a celestial body and the hour circle of a reference point in the sky instead of the Greenwich meridian because the angle thus obtained is independent of the earth's rotation. The angular distance of a body **westward** from the hour circle (upper branch) of the **first point of Aries**, measured from 0° through 360° is called **siderial hour angle**, **SHA**. The first point of Aries is the fictitious point in the sky where the sun passes through the plane of the earth's equator in spring (vernal point). The GHA of a body is the sum of the SHA of the body and the GHA of the first point of Aries, **GHA_{Aries}** :

$$GHA = SHA + GHA_{Aries}$$

(If the resulting GHA is greater than 360°, subtract 360°.)

GHA_{Aries}, measured in time units (0-24h) instead of degrees, is called **Greenwich Siderial Time**, **GST**:

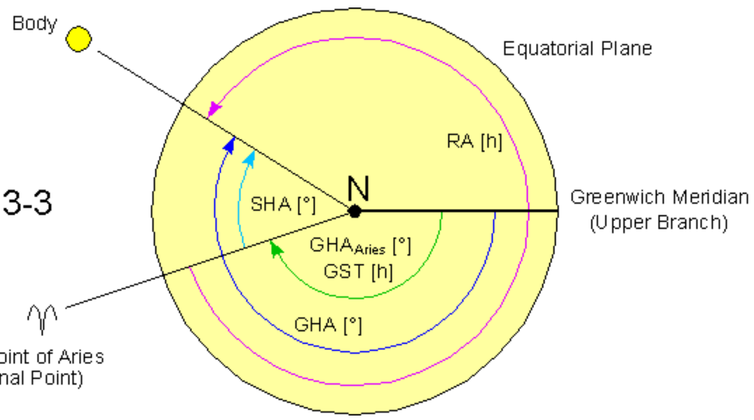
$$GST\,[h] = \frac{GHA_{Aries}\,[°]}{15} \qquad \Leftrightarrow \qquad GHA_{Aries}\,[°] = 15 \cdot GST\,[h]$$

The angular distance of a body measured in time units (0-24h) **eastward** from the hour circle of the first point of Aries is called **right ascension**, **RA**:

$$RA\,[h] = 24 - \frac{SHA\,[°]}{15} \qquad \Leftrightarrow \qquad SHA\,[°] = 360 - 15 \cdot RA\,[h]$$

Fig. 3-3 illustrates how the various hour angles are interrelated.

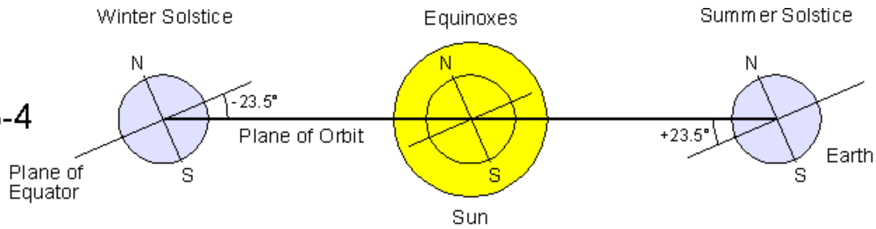
Fig. 3-3



Declinations are not affected by the rotation of the earth. The declinations of sun and planets change primarily due to the **obliquity of the ecliptic**, the inclination of the earth's equator to the plane of the earth's orbit (ecliptic). The declination of the sun, for example, varies periodically between ca. +23.5° at the time of the summer solstice and ca. -23.5° at the time of the winter solstice. At two moments during the course of a year the plane of the earth's equator passes through the center of the sun. Accordingly, the sun's declination passes through 0° (*Fig.3-4*).

When the sun is on the equator, day and night are equally long at any place on the earth. Therefore, these events are called **equinoxes** (equal nights). The apparent geocentric position of the sun in the sky at the instant of the vernal (spring) equinox marks the first point of Aries, the reference point for measuring siderial hour angles (see above).

Fig. 3-4



In addition, the declinations of the planets and the moon are influenced by the inclinations of their own orbits to the ecliptic. The plane of the moon's orbit, for example, is inclined to the ecliptic by approx. 5° and makes a tumbling movement (precession, see below) with a cycle time of 18.6 years (Saros cycle). As a result, the declination of the moon varies between approx. -28.5° and +28.5° at the beginning and at the end of the Saros cycle, and between approx. -18.5° and +18.5° in the middle of the Saros cycle.

Further, siderial hour angles and declinations of all bodies change slowly due to the influence of the **precession** of the earth's polar axis. Precession is a slow, circular movement of the polar axis along the surface of an imaginary double cone. One revolution takes about 26000 years (Platonic year). Thus, the vernal point moves along the equator at a rate of approx. 50" per year. In addition, the polar axis makes a nodding movement, called **nutation**, which causes small periodic fluctuations of the SHA's and declinations of all bodies. Last but not least, even stars are not fixed in space but have their own movements, contributing to a slow drift of their celestial coordinates.

The accurate prediction of geographic positions of celestial bodies requires complicated algorithms. Formulas for the calculation of low-precision **ephemerides** of the sun (accurate enough for celestial navigation) are given in chapter 15.

Time Measurement

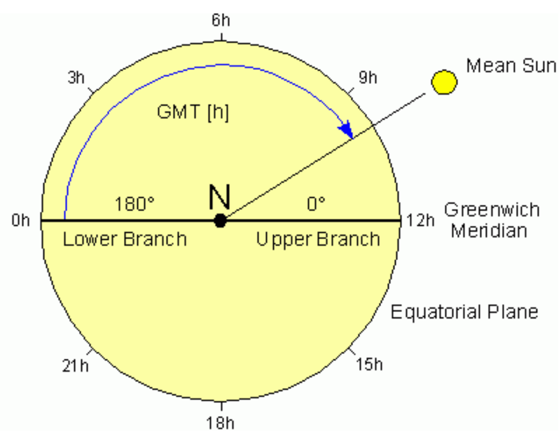
The time standard for celestial navigation is **Greenwich Mean Time**, **GMT** (now called **Universal Time**, **UT**). GMT is based upon the GHA of the (fictitious) **mean sun**:

$$GMT\,[h] = \frac{GHA_{MeanSun}\,[°]}{15} + 12$$

(If GMT is greater than 24 h, subtract 12 hours.)

In other words, GMT is the hour angle of the mean sun, expressed in hours, with respect to the lower branch of the Greenwich meridian (*Fig. 3-5*).

Fig. 3-5



By definition, the GHA of the mean sun increases by exactly 15° per hour, completing a 360° cycle in 24 hours. Celestial coordinates tabulated in the **Nautical Almanac** refer to GMT (UT).

The hourly increase of the GHA of the **apparent (observable) sun** is subject to periodic changes and is sometimes slightly greater, sometimes slightly smaller than 15° during the course of a year. This behavior is caused by the eccentricity of the earth's orbit and by the obliquity of the ecliptic. The time derived from the GHA of the apparent sun is called **Greenwich Apparent Time**, **GAT**. A sundial located at the Greenwich meridian, for example, would indicate GAT. The difference between GAT and GMT is called **equation of time, EoT**:

$$EoT = GAT - GMT$$

EoT varies periodically between approx. -16 minutes and +16 minutes. Predicted values for EoT for each day of the year (at 0:00 and 12:00 GMT) are given in the Nautical Almanac (grey background indicates negative EoT). EoT is needed when calculating times of sunrise and sunset, or determining a **noon longitude** (see chapter 6). Formulas for the calculation of EoT are given in chapter 15.

Due to the rapid change of GHA, celestial navigation requires accurate time measurement, and the time at the instant of observation should be noted to the second if possible. This is usually done by means of a chronometer and a stopwatch. The effects of time errors are dicussed in chapter 16. If GMT (UT) is not available, **UTC (Coordinated Universal Time)** can be used. UTC, based upon highly accurate atomic clocks, is the standard for radio time signals broadcast by, e. g., WWV or WWVH*. Since GMT (UT) is linked to the earth's rotating speed which decreases slowly and, moreover, with unpredictable irregularities, GMT (UT) and UTC tend to drift apart. For practical reasons, it is desirable to keep the difference between GMT (UT) and UTC sufficiently small. To ensure that the difference, DUT, never exceeds ±0.9 s, UTC is synchronized with UT by inserting or omitting leap seconds at certain times, if necessary. Current values for DUT are published by the **United States Naval Observatory**, **Earth Orientation Department**, on a regular basis (IERS Bulletin A).

$$UT = UTC + DUT$$

*It is most confusing that nowadays the term GMT is often used as a synonym for UTC instead of UT. GMT time signals from radio stations generally refer to UTC. In this publication, the term GMT is always used in the traditional (astronomical) sense, as explained above.

Terrestrial Dynamical Time, TDT, is an atomic time scale which is **not** synchronized with GMT (UT). It is a continuous and linear time measure used in astronomy (calculation of ephemerides) and space flight. TDT is presently (2001) approx. 1 minute ahead of GMT.

The Nautical Almanac

Predicted values for GHA and Dec of sun, moon and the navigational planets with reference to GMT (UT) are tabulated for each whole hour of the year on the daily pages of the **Nautical Almanac**, **N.A.**, and similar publications [12, 13]. GHA_{Aries} is tabulated in the same manner.

Listing GHA and Dec of all 57 fixed stars used in navigation for each whole hour of the year would require too much space. Since declinations of stars and (apparent) positions of stars relative to each other change only slowly, tabulated average siderial hour angles and declinations of stars for periods of 3 days are accurate enough for navigational applications.

GHA and Dec for each second of the year are obtained using the **interpolation tables** at the end of the N.A. (printed on tinted paper), as explained in the following directions:

1.

We note the exact time of observation (UTC or, preferably, UT), determined with a chronometer, for each celestial body.

2.

We look up the day of observation in the N.A. (two pages cover a period of three days).

3.

We go to the nearest whole hour preceding the time of observation and note GHA and Dec of the observed body. In case of a fixed star, we form the sum of GHA Aries and the SHA of the star, and note the average Dec. When observing sun or planets, we note the ***v*** and ***d*** factors given at the bottom of the appropriate column. For the moon, we take *v* and *d* for the nearest whole hour preceding the time of observation.

The quantity *v* is necessary to apply an additional correction to the following interpolation of the GHA of moon and planets. It is not required for stars. The sun does not require a *v* factor since the correction has been incorporated in the tabulated values for the sun's GHA.

The quantity *d*, which is negligible for stars, is the change of Dec during the time interval between the nearest whole hour preceding the observation and the nearest whole hour following the observation. It is needed for the interpolation of Dec.

4.

We look up the minute of observation in the interpolation tables (1 page for each 2 minutes of the hour), go to the second of observation, and note the **increment** from the appropriate column.

We enter one of the three columns to the right of the increment columns with the *v* and *d* factors and note the corresponding **corr**(ection) values (*v*-corr and *d*-corr).

The sign of *d*-corr depends on the trend of declination at the time of observation. It is positive if Dec at the whole hour following the observation is greater than Dec at the whole hour preceding the observation. Otherwise it is negative.

V-corr is negative for Venus and otherwise always positive.

5.

We form the sum of Dec and *d*-corr (if applicable).

We form the sum of GHA (or GHA Aries and SHA of star), increment, and *v*-corr (if applicable).

Interactive Computer Ephemeris

The **Interactive Computer Ephemeris**, **ICE**, developed by the U.S. Naval Observatory, is a DOS program (successor of the **Floppy Almanac**) for the calculation of ephemeral data for sun, moon, planets and stars.

ICE is FREEWARE (no longer supported by USNO), compact, easy to use, and provides a vast quantity of accurate astronomical data for a time span of almost 250 (!) years.

Among many other features, ICE calculates GHA and Dec for a given body and time as well as altitude and azimuth of the body for an **assumed position** (see chapter 4) and sextant altitude corrections. Since the calculated data are as accurate as those tabulated in the **Nautical Almanac** (approx. 0.1'), the program makes an adequate alternative, although a printed almanac (and sight reduction tables) should be kept as a backup in case of a computer failure.

The following instructions refer to the final version (0.51). Only program features relevant to navigation are explained.

1. Installation
Copy the program files to a chosen directory on the hard drive or to a floppy disk.

2. Getting Started
Change to the program directory (or floppy disk) and enter "ice". The main menu appears. Use the function keys F1 to F10 to navigate through the submenus. The program is more or less self-explanatory.

Go to the submenu INITIAL VALUES (F1). Follow the directions on the screen to enter date and time of observation (F1), assumed latitude (F2), assumed longitude (F3), and your local time zone (F6). Assumed latitude and longitude define your assumed position. Use the correct data format, as shown on the screen (decimal format for latitude and longitude). After entering the above data, press F7 to accept the values displayed.

To change the default values permanently, edit the file ice.dft with a text editor (after making a backup copy) and make the appropriate changes. Do not change the data format. The numbers have to be in columns 21-40.

An output file can be created to store calculated data. Go to the submenu FILE OUTPUT (F2) and enter a chosen file name, e.g., OUTPUT.TXT.

3. Calculation of Navigational Data
From the main menu, go to the submenu NAVIGATION (F7). Enter the name of the body. The program displays GHA and Dec of the body, GHA and Dec of the sun (if visible), and GHA of the vernal equinox for the time (UT) stored in INITIAL VALUES. Hc (computed altitude) and Zn (azimuth) mark the apparent position of the body as observed from the assumed position. Approximate altitude corrections (refraction, SD, PA), based upon Hc, are also displayed (for lower limb of body). The semidiameter of the moon includes augmentation. The coordinates calculated for Venus and Mars do **not** include phase correction. Therefore, the upper or lower limb (if visible) should be observed. ΔT is TDT-UT, the difference between **terrestrial dynamical time** and UT for the date given (presently approx. 1 min.).

Horizontal parallax and semidiameter of a body can be extracted indirectly, if required, from the submenu POSITIONS (F3). Choose APPARENT GEOCENTRIC POSITIONS (F1) and enter the name of the body (sun, moon, planets). The last column shows the distance of the center of the body from the center of the earth, measured in astronomical units (1 AU = 149.6 · 10^6 km). HP and SD are calculated as follows:

$$HP = \arcsin \frac{r_E\,[km]}{distance\,[km]} \qquad\qquad SD = \arcsin \frac{r_B\,[km]}{distance\,[km]}$$

r_E is the equatorial radius of the earth (6378 km). r_B is the radius of the body (Sun: 696260 km, Moon: 1378 km, Venus: 6052 km, Mars: 3397 km, Jupiter: 71398 km, Saturn: 60268 km).

The apparent geocentric positions refer to TDT, but the difference between TDT and UT has no significant effect on HP and SD.

To calculate times of rising and setting of a body, go to the submenu RISE & SET TIMES (F6) and enter the name of the body. The columns on the right display the time of rising, meridian transit, and setting for your assumed location (UT+xh, according to the time zone specified).

Multiyear Interactive Computer Almanac

The **Multiyear Interactive Computer Almanac**, **MICA**, is the successor of ICE. MICA 1.5 includes the time span from 1990 through 2005. Versions for DOS and Macintosh are on one CD-ROM. MICA provides highly accurate ephemerides primarily for astronomical applications.

For navigational purposes, zenith distance and azimuth of a body with respect to an assumed position can also be calculated.

MICA computes RA and Dec but not GHA. Since MICA calculates GST, GHA can be obtained by applying the formulas shown at the beginning of the chapter. The following instructions refer to the DOS version.

Right ascension and declination of a body can be accessed through the following menus and submenus:
Calculate
Positions
Objects (choose body)
Apparent
Geocentric
Equator of Date

Greenwich siderial time is accessed through:
Calculate
Time & Orientation
Siderial Time
(App.)

The knowledge of corrected altitude and geographic position of a body enables the navigator to establish a **line of position**, as will be explained in chapter 4.

Chapter 4

Finding One's Position (Sight Reduction)

Lines of Position

Any geometrical or physical line passing through the observer's (still unknown) position and accessible through measurement or observation is called a **line of position**, **LoP**. Examples are circles of equal altitude, meridians of longitude, parallels of latitude, bearing lines (compass bearings) of terrestrial objects, coastlines, rivers, roads, or railroad tracks. **A single LoP indicates an infinite series of possible positions. The observer's actual position is marked by the point of intersection of at least two LoP's, regardless of their nature.** The concept of the position line is essential to modern navigation.

Sight Reduction

Deriving a line of position from altitude and GP of a celestial object is called **sight reduction** in navigator´s language. Understanding the process completely requires some background in spherical trigonometry, but knowing the basic concepts and a few equations is sufficient for most applications of celestial navigation. The theoretical explanation, using the **law of cosines** and the **navigational triangle**, is given in chapter 10 and 11. In the following, we will discuss the semi-graphic methods developed by *Sumner* and *St. Hilaire*. Both methods require relatively simple calculations only and enable the navigator to plot lines of position on a navigation chart or **plotting sheet** (see chapter 13).

Knowing altitude and GP of a body, we also know the radius of the corresponding circle of equal altitude (our line of position) and the location of its center. As mentioned in chapter 1 already, plotting circles of equal altitude directly on a chart is usually impossible due to their large dimensions and the distortions caused by map projection. However, *Sumner* and *St. Hilaire* showed that only a small arc of each circle of equal altitude is needed to find one's position. Since this arc is comparatively short, it can be replaced with a secant or tangent a of the circle.

The Intercept Method

This is the most versatile and most popular sight reduction procedure. In the second half of the 19th century, the French navy officer and later admiral *St. Hilaire* found that a straight line **tangent** to the circle of equal altitude in the vicinity of the observer's position can be utilized as a line of position. The procedure comprises the following steps:

1.

First, we need an **initial position** which should be less than ca. 100 nm away from our actual (unknown) position. This may be our **estimated position**, our **dead reckoning position**, **DRP** (chapter 11), or an **assumed position**, **AP**. We mark this position on our navigation chart or **plotting sheet** (chapter 13) and note the corresponding latitude and longitude. An assumed position is a chosen point in the vicinity of our estimated position or DRP, preferably the nearest point on the chart where two grid lines intersect. An assumed position is sometimes preferred since it may be more convenient for plotting lines and measuring angles on the plotting sheet. Some sight reduction tables are based upon AP's because they require integer values for coordinates. **The following procedures and formulas refer to an AP. They would be exactly the same, however, when using a DRP or an estimated position.**

2.

Using the laws of spherical trigonometry (chapter 10 and 11), we calculate the altitude of the observed body as it would appear at our AP (reduced to the celestial horizon). This altitude is called **calculated** or **computed altitude**, **Hc**:

$$Hc = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA\right)$$

or

$$Hc = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos t\right)$$

Lat_{AP} is the **geographic latitude** of AP. **Dec** is the declination of the observed body. **LHA** is the **local hour angle** of the body, the angular distance of GP westward from the local meridian going through AP, measured from 0° through 360°.

$$LHA = \begin{cases} GHA + Lon_{AP} & \text{if} \quad 0° \leq GHA + Lon_{AP} \leq 360° \\ GHA + Lon_{AP} + 360° & \text{if} \quad GHA + Lon_{AP} < 0° \\ GHA + Lon_{AP} - 360° & \text{if} \quad GHA + Lon_{AP} > 360° \end{cases}$$

Instead of the local hour angle, we can use the **meridian angle**, **t**, to calculate Hc. Like LHA, t is the algebraic sum of GHA and Lon_{AP}. In contrast to LHA, however, t is measured westward (0°...+180°) or eastward (0°...–180°) from the local meridian:

$$t = \begin{cases} GHA + Lon_{AP} & \text{if} \quad GHA + Lon_{AP} \leq 180° \\ GHA + Lon_{AP} - 360° & \text{if} \quad GHA + Lon_{AP} > 180° \end{cases}$$

Lon_{AP} is the **geographic longitude** of AP. **The sign of Lon_{AP} has to be observed carefully (E:+, W:–).**

3.

We calculate the **azimuth** of the body, Az_N, the direction of GP with reference to the geographic north point on the horizon, measured clockwise from 0° through 360° at AP. We can calculate the azimuth either from Hc (altitude azimuth) or from LHA or t (time azimuth). Both methods give identical results.

The formula for the altitude azimuth is stated as:

$$Az = \arccos \frac{\sin Dec - \sin Hc \cdot \sin Lat_{AP}}{\cos Hc \cdot \cos Lat_{AP}}$$

The **azimuth angle**, **Az**, the angle formed by the meridian going through AP and the great circle going through AP and GP, is not necessarily identical with Az_N since the arccos function yields results between 0° and +180°. To obtain Az_N, we apply the following rules:

$$Az_N = \begin{cases} Az & \text{if} \quad \sin LHA \leq 0 \quad (\text{or} \quad t \leq 0) \\ 360° - Az & \text{if} \quad \sin LHA > 0 \quad (\text{or} \quad t > 0) \end{cases}$$

The formula for the time azimuth is stated as:

$$Az = \arctan \frac{-\sin LHA}{\cos Lat_{AP} \cdot \tan Dec - \sin Lat_{AP} \cdot \cos LHA}$$

Again, the meridian angle, t, may be substituted for LHA. Since the arctan function returns results between -90° and +90°, the time azimuth formula requires a different set of rules to obtain Az_N:

$$Az_N = \begin{cases} Az & \text{if} \quad \text{numerator} > 0 \quad \text{AND} \quad \text{denominator} > 0 \\ Az + 360° & \text{if} \quad \text{numerator} < 0 \quad \text{AND} \quad \text{denominator} > 0 \\ Az + 180° & \text{if} \quad \text{denominator} < 0 \end{cases}$$

Fig. 4-1 illustrates the angles involved in the calculation of Hc (= 90°-z) and Az:

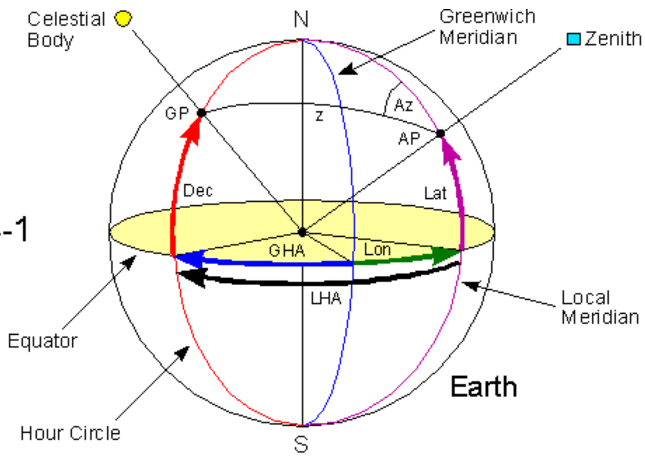


Fig. 4-1

The above formulas are derived from the **navigational triangle** formed by N, AP, and GP. A detailed explanation is given in chapter 11. Mathematically, the calculation of Hc and Az_N is a transformation of equatorial coordinates to horizontal coordinates.

4.

We calculate the **intercept**, **Ic**, the difference between observed altitude, Ho, (chapter 2) and computed altitude, Hc. For the following procedures, the intercept, which is directly proportional to the difference between the radii of the corresponding circles of equal altitude, is expressed in distance units:

$$Ic[nm] = 60 \cdot (Ho[°] - Hc[°]) \qquad \text{or} \qquad Ic[km] = \frac{40031.6}{360} \cdot (Ho[°] - Hc[°])$$

(The mean perimeter of the earth is 40031.6 km.)

When going the distance Ic along the azimuth line from AP toward GP (Ic > 0) or away from GP (Ic < 0), we reach the circle of equal altitude for our actual position (LoP). As shown in *Fig. 4-2*, a straight line perpendicular to the azimuth line and tangential to the circle of equal altitude for the actual position is a fair approximation of our circular LoP as long as we stay in the vicinity of our position.

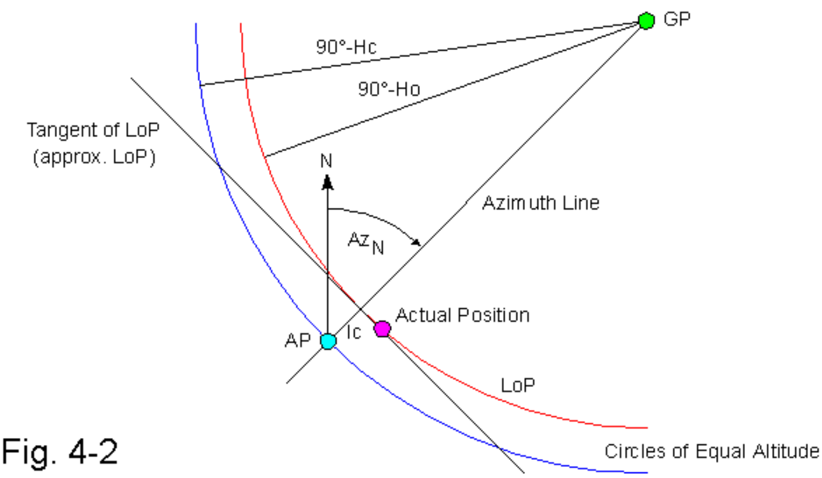


Fig. 4-2

5.

We take the chart and draw a suitable part of the azimuth line through AP. We measure the intercept, Ic, along the azimuth line (towards GP if Ic>0, away from GP if Ic<0) and draw a perpendicular through the point thus located. This perpendicular is our approximate line of position (*Fig. 4-3*).

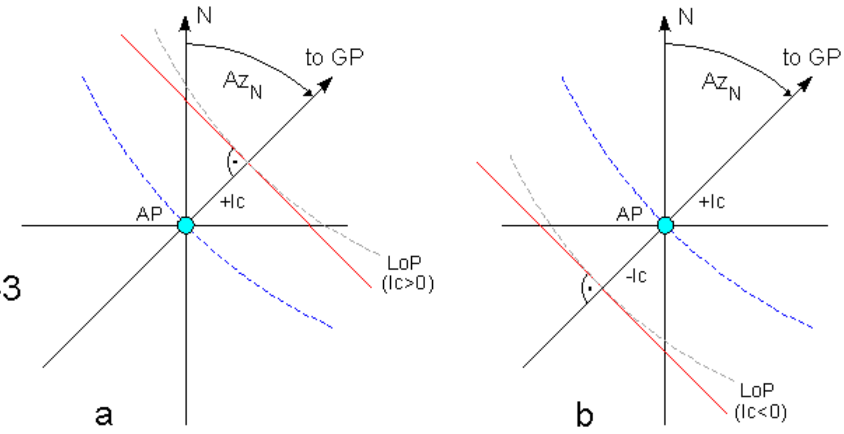


Fig. 4-3

6.

To obtain the second LoP needed to find our position, we repeat the procedure (same AP) with altitude and GP of a second celestial body or the same body at a different time of observation (*Fig. 4-4*). The point where both LoP's (tangents) intersect is our **improved position**. In navigator's language, the position thus located is called **fix**.

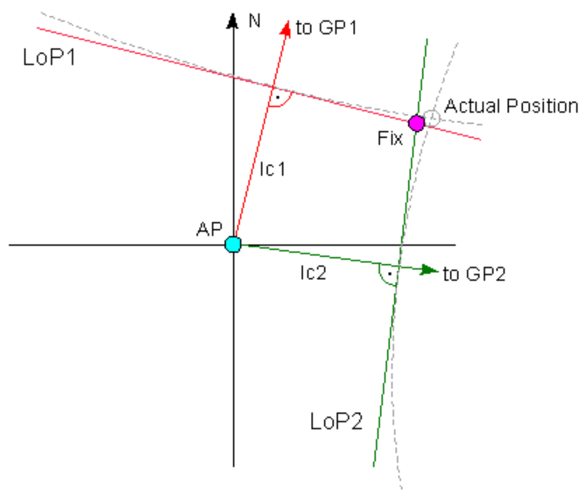


Fig. 4-4

The intercept method ignores the curvature of the actual LoP's. The resulting error remains tolerable as long as the radii of the circles of equal altitude are great enough and AP is not too far from the actual position (see chapter 16). The geometric error inherent to the intercept method can be decreased by **iteration**, i.e., substituting the position thus obtained for AP and repeating the calculations (same altitudes and GP's). This will result in a more accurate position. If necessary, we can reiterate the procedure until the obtained position remains virtually constant.

Since a dead reckoning position is usually nearer to our true position than an assumed position, the latter may require a greater number of iterations.

Accuracy is also improved by observing three bodies instead of two. Theoretically, the LoP's should intersect each other at a single point. Since no observation is entirely free of errors, we will usually obtain three points of intersection forming an **error triangle** (*Fig. 4-5*).

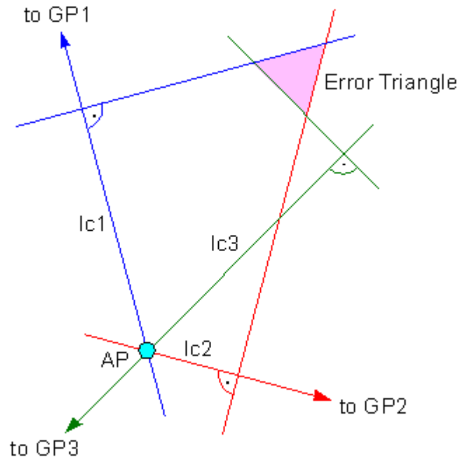


Fig. 4-5

Area and shape of the triangle give us a rough estimate of the quality of our observations (see chapter 16). Our **most probable position**, **MPP**, is usually in the vicinity of the center of the inscribed circle of the triangle (the point where the bisectors of the three angles meet).

When observing more than three bodies, the resulting LoP's will form the corresponding polygons.

Direct Computation

If we do not want to plot our lines of position to determine our fix, we can find the latter by computation. Using the **method of least squares**, it is possible to calculate the most probable position directly from an unlimited number, n, of observations (n > 1) without the necessity of a graphic plot. The Nautical Almanac provides the following procedure. First, the auxiliary quantities A, B, C, D, E, and G have to be calculated:

$$A = \sum_{i=1}^{n} \cos^2 Az_i \qquad B = \sum_{i=1}^{n} \sin Az_i \cdot \cos Az_i \qquad C = \sum_{i=1}^{n} \sin^2 Az_i$$

$$D = \sum_{i=1}^{n} Ic_i \cdot \cos Az_i \qquad E = \sum_{i=1}^{n} Ic_i \cdot \sin Az_i \qquad G = A \cdot C - B^2$$

The geographic coordinates of the observer's MPP are then obtained as follows:

$$Lon = Lon_{AP} + \frac{A \cdot E - B \cdot D}{G \cdot \cos Lat_{AP}} \qquad Lat = Lat_{AP} + \frac{C \cdot D - B \cdot E}{G}$$

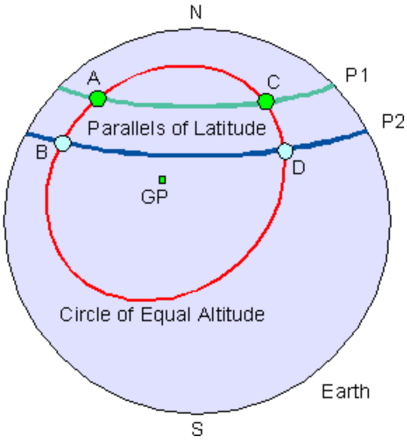
The method does not correct the geometric errors inherent to the intercept method. These are eliminated, if necessary, by applying the method iteratively until the MPP remains virtually constant. The N.A. suggests repeating the calculations if the obtained MPP is more than 20 nautical miles from AP or the initial estimated position.

Sumner's Method

This sight reduction procedure was discovered by *T. H. Sumner*, an American sea captain, in the first half of the 19th century. Although it is rarely used today, it is still an interesting alternative to *St. Hilaire's* intercept method. The theoretical explanation is given in chapter 11 (navigational triangle).

Sumner had the brilliant idea to derive a line of position from the points where a circle of equal altitude intersects two chosen parallels of latitude, P1 and P2 (*Fig. 4-6*).

Fig. 4-6



An observer being between the parallels P1 and P2 is either on the arc A-B or on the arc C-D. With an estimate of his longitude, the observer can easily find on which of both arcs he is, for example, A-B. The arc thus found is the relevant part of his line of position, the other arc is discarded. We can approximate the LoP by drawing a straight line through A and B which is a secant of the circle of equal altitude. This secant is called **Sumner line**. Before plotting the Sumner line on our chart, we have to find the longitudes of the points of intersection, A, B, C, and D. This is the procedure:

1.

We choose a parallel of latitude (P1) north of our estimated latitude. Preferably, the assumed latitude, Lat, should refer to the nearest grid line on our chart or plotting sheet.

2.

Solving the altitude formula (see above) for t and substituting Ho for Hc, we get:

$$t = \pm \arccos \frac{\sin Ho - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

Now, t is a function of latitude, declination, and the observed altitude of the body. Lat is the assumed latitude. In other words, the meridian angle of a body is either +t or –t when an observer being at the latitude Lat measures the altitude Ho. Using the following formulas, we obtain the longitudes which mark the points where the circle of equal altitude intersects the assumed parallel of latitude, for example, the points A and C if we choose P1:

$$Lon_1 = t - GHA$$

$$\text{If } Lon_1 < -180° \quad \rightarrow \quad Lon_1 + 360°$$

$$Lon_2 = 360° - t - GHA$$

$$\text{If } Lon_2 < -180° \quad \rightarrow \quad Lon_2 + 360°$$
$$\text{If } Lon_2 > +180° \quad \rightarrow \quad Lon_2 - 360°$$

Comparing the longitudes thus obtained with our estimated longitude, we select the relevant longitude and discard the other. This method of finding longitude is called **time sight** (see chapter 6).

3.

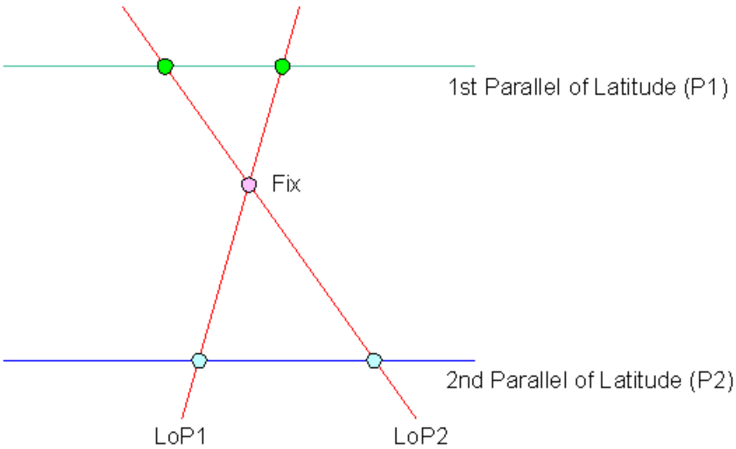
We chose a parallel of latitude (P2) south of our estimated latitude. The distance between P1 and P2 should not exceed 1 or 2 degrees. We repeat steps 1 and 2 with the second parallel of latitude, P2.

4.

On our plotting sheet, we mark each remaining longitude on the corresponding parallel and plot the Sumner line through the points thus located.

To obtain a fix, we repeat the above procedure with the same parallels and a second body. The point where both Sumner lines, LoP1 and LoP2, intersect is our fix (*Fig. 4-7*).

Fig. 4-7



If both assumed parallels of latitude are either north or south of our actual position, we will of course find the point of intersection outside the interval defined by both parallels. Nevertheless, a fix thus obtained is correct.

A fix obtained with *Sumner's* method, too, has a small error caused by neglecting the curvature of the circles of equal altitude. Similar to the intercept method, we can improve the fix by iteration. In this case, we choose a new pair of assumed latitudes, nearer to the fix, and repeat the whole procedure.

A Sumner line may be inaccurate under certain conditions (see time sight, chapter 6). Apart from these restrictions, *Sumner's* method is fully adequate. It has even the advantage that lines of position are plotted without a protractor.

As with the intercept method, we can plot Sumner lines resulting from three (or more) observations to obtain an error triangle (polygon).

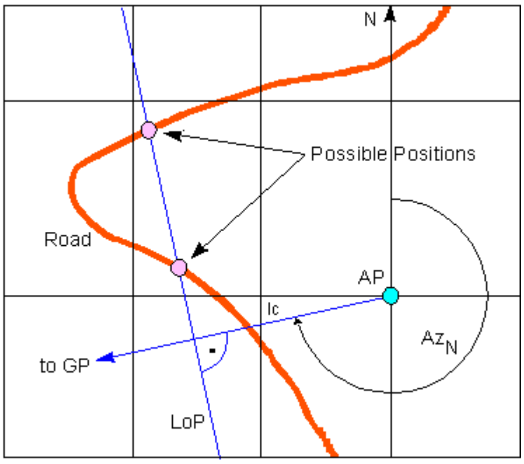
Sumner's method revolutionized celestial navigation and can be considered as the beginning of modern position line navigation which was later perfected by *St. Hilaire's* intercept method.

Combining Different Lines of Position

Since the point of intersection of any two LoP's, regardless of their nature, marks the observer's geographic position, one celestial LoP may suffice to find one's position if another LoP of a different kind is available.

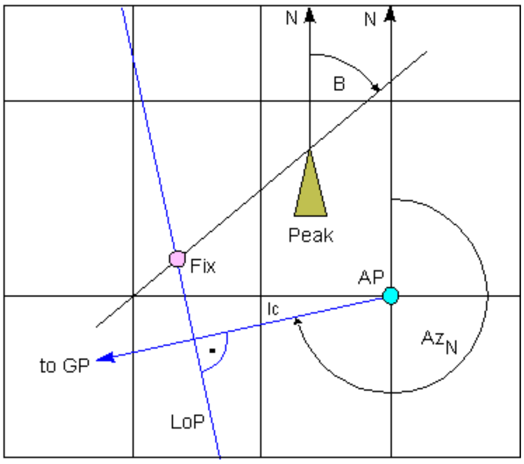
In the desert, for instance, we can determine our current position by finding the point on the map where a LoP obtained by observation of a celestial object intersects the dirt road we are traveling on (*Fig. 4-8*).

Fig. 4-8



We could as well find our position by combining our celestial LoP with the bearing line of a distant mountain peak or any other prominent landmark (*Fig. 4-9*). B is the compass bearing of the terrestrial object (corrected for magnetic declination).

Fig. 4-9



Both examples clearly demonstrate the versatility of position line navigation.

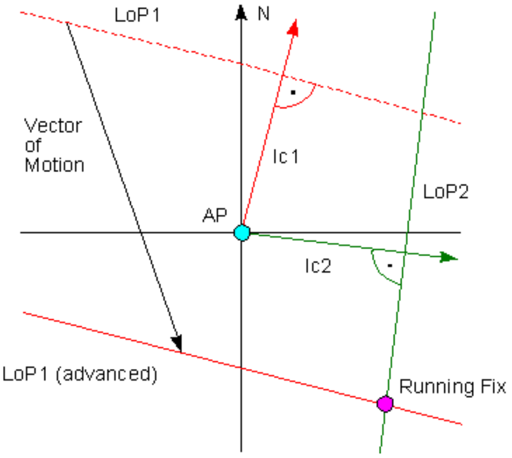
Chapter 5

Finding the Position of a Traveling Vessel

The intercept method even enables the navigator to determine the position of a vessel traveling a considerable distance between two observations, provided course and speed over ground are known.

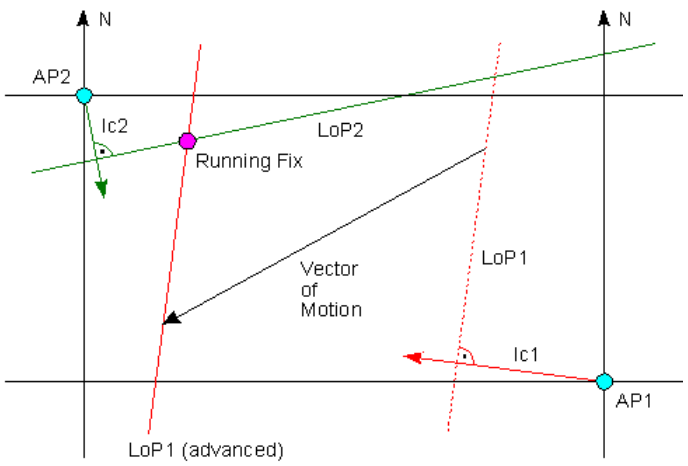
We begin with plotting both lines of position in the usual manner, as illustrated in chapter 4, *Fig. 4-4*. Then, we apply the vector of motion (defined by course, speed, and time elapsed) to the LoP resulting from the first observation, and plot the **advanced** first LoP, the parallel of the first LoP thus obtained. **The point where the advanced first LoP intersects the second LoP is the position of the vessel at the time of the second observation.** A position obtained in this fashion is called a **running fix** (*Fig. 5-1*).

Fig. 5-1



The procedure gives good results when traveling short distances (up to approx. 30 nm) between the observations. When traveling a larger distance (up to approx. 150 nm), it may be necessary to choose two different AP's, not too far away from each estimated position, to reduce geometric errors (*Fig. 5-2*).

Fig.5-2



It is also possible to find the running fix for the time of the first observation. In this case the second LoP has to be **retarded** (moved backwards).

Sumner lines and terrestrial lines of position may be advanced or retarded in the same manner.

In practice, course and speed over ground can only be estimated since the exact effects of currents and wind are usually not known. Therefore, a running fix is usually not as accurate as a stationary fix.

Chapter 6

Methods for Latitude and Longitude Measurement

Latitude by Polaris

The observed altitude of a star being vertically above the geographic north pole would be numerically equal to the latitude of the observer (*Fig. 6-1*).

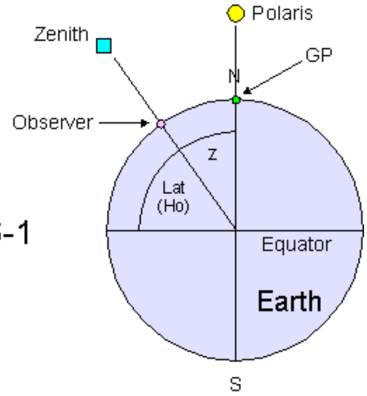


Fig. 6-1

This is nearly the case with the pole star (Polaris). However, since there is a measurable angular distance between Polaris and the polar axis of the earth (presently ca. 1°), the altitude of Polaris is a function of LHA_{Aries}. Nutation, too, influences the altitude of Polaris measurably. To obtain the accurate latitude, several corrections have to be applied:

$$Lat = Ho - 1° + a_0 + a_1 + a_2$$

The corrections a_0, a_1, and a_2 depend on LHA_{Aries}, the observer's estimated latitude, and the number of the month. They are given in the Polaris Tables of the Nautical Almanac [12]. To extract the data, the observer has to know his approximate position and the approximate time.

Noon Latitude (Latitude by Maximum Altitude)

This is a very simple method enabling the observer to determine his latitude by measuring the maximum altitude of an object, particularly the sun. No accurate time measurement is required. The altitude of the sun passes through a flat maximum **approximately** (see noon longitude) at the moment of upper meridian passage (**local apparent noon, LAN**) when the GP of the sun has the same longitude as the observer and is either north or south of him, depending on the observer's geographic latitude. The observer's latitude is easily calculated by forming the algebraic sum or difference of declination and observed zenith distance z (90°-Ho) of the sun. depending on whether the sun is north or south of the observer (*Fig. 6-2*).

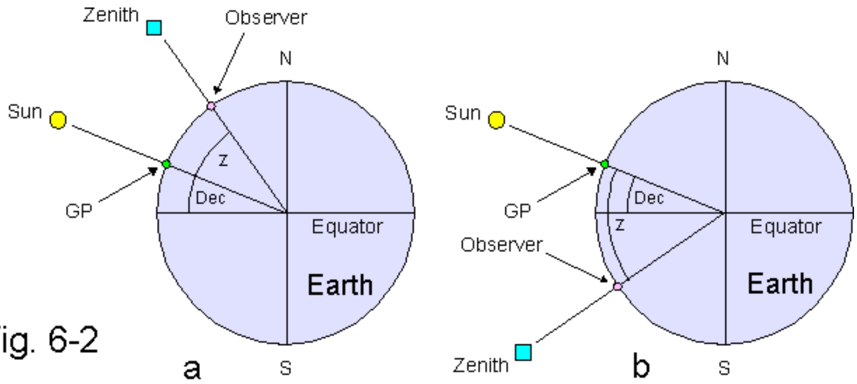


Fig. 6-2

1. Sun south of observer (*Fig. 6-2a*): $$Lat = Dec + (90° - Ho)$$

2. Sun north of observer (*Fig. 6-2b*): $$Lat = Dec - (90° - Ho)$$

Northern declination is positive, southern negative.

Before starting the observations, we need a rough estimate of our current longitude to know the time (GMT) of LAN. We look up the time of Greenwich meridian passage of the sun on the daily page of the Nautical Almanac and add 4 minutes for each degree western longitude or subtract 4 minutes for each degree eastern longitude. To determine the maximum altitude, we start observing the sun approximately 15 minutes before LAN. We follow the increasing altitude of the sun with the sextant, note the maximum altitude when the sun starts descending again, and apply the usual corrections.

We look up the declination of the sun at the approximate time (GMT) of local meridian passage on the daily page of the Nautical Almanac and apply the appropriate formula.

Historically, noon latitude and latitude by Polaris are among the oldest methods of celestial navigation.

Ex-Meridian Sight

Sometimes, it may be impossible to measure the maximum altitude of the sun. For example, the sun may be obscured by a cloud at this moment. If we have a chance to measure the altitude of the sun a few minutes before or after meridian transit, we are still able to find our exact latitude by reducing the observed altitude to the meridian altitude, provided we know our exact longitude (see below) and have an estimate of our latitude.

First, we need the time of local meridian transit (eastern longitude is positive, western longitude negative):

$$T_{Transit}[GMT] = 12 - EoT[h] - \frac{Lon[°]}{15}$$

The meridian angle of the sun, t, is calculated from the time of observation:

$$t[°] = 15 \cdot (T_{Observation}[GMT] - T_{Transit}[GMT])$$

Starting with our estimated Latitude, Lat_E, we calculate the altitude of the sun at the time of observation. We use the altitude formula from chapter 4:

$$Hc = \arcsin(\sin Lat_E \cdot \sin Dec + \cos Lat_E \cdot \cos Dec \cdot \cos t)$$

We further calculate the altitude of the sun at meridian transit, H_{MTC}:

$$H_{MTC} = 90° - |Lat_E - Dec|$$

The difference between H_{MTC} and Hc is called **reduction**, **R**:

$$R = H_{MTC} - Hc$$

Adding R to the **observed altitude**, **Ho**, we get approximately the altitude we would observe at meridian transit, H_{MTO}:

$$H_{MTO} \approx Ho + R$$

From H_{MTO}, we can calculate our improved latitude, $Lat_{improved}$:

$$Lat_{improved} = Dec \pm (90° - H_{MTO})$$

(sun south of observer: +, sun north of observer: –)

The exact latitude is obtained by iteration, i. e., we substitute $Lat_{improved}$ for Lat_E and repeat the calculations until the obtained latitude is virtually constant. Usually, no more than one or two iterations are necessary. The method has a few limitations and requires critical judgement. The meridian angle should be smaller than about one quarter of the expected zenith distance at meridian transit ($z_{MT} = |Lat_E - Dec|$), and the meridian zenith distance should be at least four times greater than the estimated error of Lat_E. Otherwise, a greater number of iterations may be necessary. Dec must **not** lie between Lat_E and the true latitude because the method yields erratic results in such cases. If in doubt, we can calculate with different estimated latitudes and compare the results. For safety reasons, the sight should be discarded if the meridian altitude exceeds approx. 85°. If t is not a small angle (t > 1°), we may have to correct the latitude last found for the change in declination between the time of observation and the time of meridian transit, depending on the current rate of change of Dec.

Noon Longitude (Longitude by Equal Altitudes, Longitude by Meridian Transit)

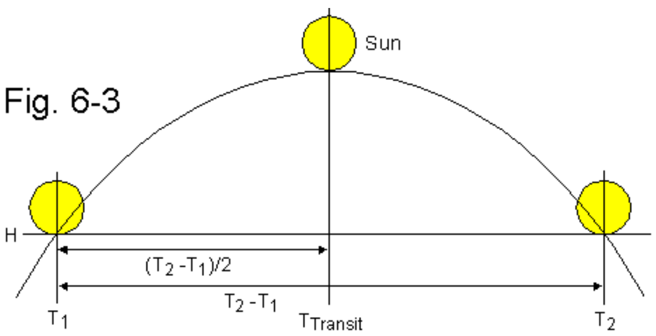
Since the earth rotates with an angular velocity of 15° per hour with respect to the mean sun, the time of local meridian transit (local apparent noon) of the sun, $T_{Transit}$, can be used to calculate the observer's longitude:

$$Lon[°] = 15 \cdot (12 - T_{Transit}[h] - EoT_{Transit}[h])$$

$T_{Transit}$ is measured as GMT (decimal format). The correction for EoT at the time of meridian transit, $EoT_{Transit}$, has to be made because the apparent sun, not the mean sun, is observed (see chapter 3). Since the Nautical Almanac contains only values for EoT (see chapter 3) at 0:00 GMT and 12:00 GMT of each day, $EoT_{Transit}$ has to be found by interpolation.

Since the altitude of the sun - like the altitude of any celestial body - passes through a rather flat maximum, the time of peak altitude is difficult to measure. The exact time of meridian transit can be derived, however, from two equal altitudes of the sun.

Assuming that the sun moves along a symmetrical arc in the sky, $T_{Transit}$ is the mean of the times corresponding with a chosen pair of equal altitudes of the sun, one occurring before LAN (T_1), the other past LAN (T_2) (*Fig. 6-3*):



$$T_{Transit} = \frac{T_1 + T_2}{2}$$

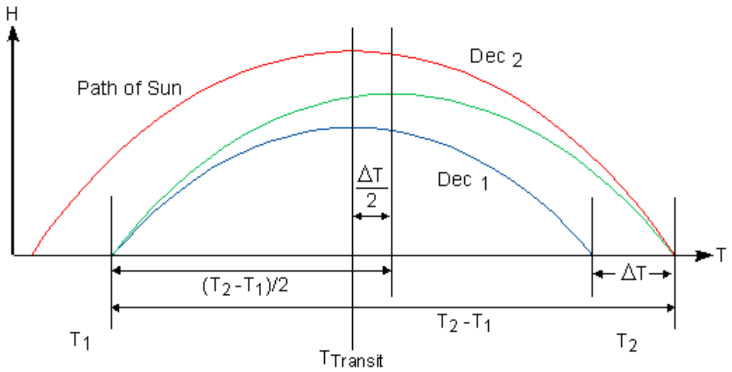
In practice, the times of two equal altitudes of the sun are measured as follows:

In the morning, the observer records the time (T_1) corresponding with a chosen altitude, H. In the afternoon, the time (T_2) is recorded when the descending sun passes through the same altitude again. Since only times of equal altitudes are measured, no altitude correction is required. The interval T_2-T_1 should be greater than 1 hour.

Unfortunately, the arc of the sun is only symmetrical with respect to $T_{Transit}$ if the sun's declination is fairly constant during the observation interval. This is approximately the case around the times of the solstices.

During the rest of the year, particularly at the times of the equinoxes, $T_{Transit}$ differs significantly from the mean of T_1 and T_2 due to the changing declination of the sun. *Fig. 6-4* shows the altitude of the sun as a function of time and illustrates how the changing declination affects the apparent path of the sun in the sky.

Fig. 6-4



The blue line shows the path of the sun for a given, constant declination, Dec_1. The red line shows how the path would look with a different declination, Dec_2. In both cases, the apparent path of the sun is symmetrical with respect to $T_{Transit}$. However, if the sun's declination varies from Dec_1 at T_1 to Dec_2 at T_2, the path shown by the green line will result. Now, the times of equal altitudes are no longer symmetrical to $T_{Transit}$. The sun's meridian transit occurs before $(T_2+T_1)/2$ if the sun's declination changes toward the observer's parallel of latitude, like shown in *Fig. 6-4*. Otherwise, the meridian transit occurs after $(T_2+T_1)/2$. Since time and local hour angle (or meridian angle) are proportional to each other, a systematic error in longitude results.

The error in longitude is negligible around the times of the solstices when Dec is almost constant, and is greatest (up to several arcminutes) at the times of the equinoxes when the rate of change of Dec is greatest (approx. 1 arcminute per hour). Moreover, the error in longitude increases with the observer's latitude and may be quite dramatic in polar regions.

The obtained longitude can be improved, if necessary, by application of the **equation of equal altitudes** [5]:

$$\Delta t \approx \left(\frac{\tan Lat}{\sin t_2} - \frac{\tan Dec_2}{\tan t_2} \right) \cdot \Delta Dec \qquad\qquad t_2[°] \approx 15 \cdot \frac{(T_2[h] - T_1[h])}{2}$$

t_2 is the meridian angle of the sun at T_2. Δt is the change in t which cancels the change in altitude resulting from the change in declination between T_1 and T_2, ΔDec.

Lat is the observer's latitude, e. g., a noon latitude. If no accurate latitude is available, an estimated latitude may be used. Dec_2 is the declination of the sun at T_2.

The corrected second time of equal altitude, T_2^*, is:

$$T_2^*[h] = T_2[h] - \Delta T[h] = T_2[h] - \frac{\Delta t[°]}{15}$$

At T_2^*, the sun would pass through the same altitude as measured at T_1 if Dec did not change during the interval of observation. Accordingly, the time of meridian transit is:

$$T_{Transit} = \frac{T_1 + T_2^*}{2}$$

The correction is very accurate if the exact value for ΔDec is known. Calculating ΔDec with MICA yields a more reliable correction than extracting ΔDec from the Nautical Almanac. If no precise computer almanac is available, ΔDec should be calculated from the daily change of declination to keep the rounding error as small as possible.

Although the equation of equal altitudes is strictly valid only for an infinitesimal change of Dec, dDec, it can be used for a measurable change, ΔDec, (up to several arcminutes) as well without sacrificing much accuracy. Accurate time measurement provided, the residual error in longitude should be smaller than ±0.1' in most cases.

The above formulas are not only suitable to determine one's exact longitude but can also be used to determine the chronometer error if one's exact position is known. This is done by comparing the time of meridian transit calculated from one's longitude with the time of meridian transit derived from the observation of two equal altitudes.

Fig. 6-5 shows that the maximum altitude of the sun is slightly different from the altitude at the moment of meridian passage if the declination changes. Since the sun's hourly change of declination is never greater than approx. 1' and since the maximum of altitude is rather flat, the resulting error of a noon latitude is not significant (see end of chapter).

The equation of equal altitudes is derived from the altitude formula (see chapter 4) using **differential calculus**:

$$\sin H = \sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos t$$

First, we want to know how a small change in declination would affect sin H. We differentiate sin H with respect to Dec:

$$\frac{\partial(\sin H)}{\partial Dec} = \sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t$$

Thus, the change in sin H caused by an infinitesimal change in declination, d Dec, is:

$$\frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec = (\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t) \cdot d\,Dec$$

Now we differentiate sin H with respect to t in order to find out how a small change in the meridian angle would affect sin H:

$$\frac{\partial(\sin H)}{\partial t} = -\cos Lat \cdot \cos Dec \cdot \sin t$$

The change in sin H caused by an infinitesimal change in the meridian angle, dt, is:

$$\frac{\partial(\sin H)}{\partial t} \cdot d\,t = -\cos Lat \cdot \cos Dec \cdot \sin t \cdot d\,t$$

Since we want both effects to cancel each other, the total differential has to be zero:

$$\frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec + \frac{\partial(\sin H)}{\partial t} \cdot d\,t = 0$$

$$-\frac{\partial(\sin H)}{\partial t} \cdot d\,t = \frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec$$

$$\cos Lat \cdot \cos Dec \cdot \sin t \cdot d\,t \;=\; \left(\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t\right) \cdot d\,Dec$$

$$d\,t \;=\; \frac{\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot d\,Dec$$

$$d\,t \;=\; \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t}\right) \cdot d\,Dec$$

Longitude Measurement on a Traveling Vessel

On a traveling vessel, we have to take into account not only the influence of varying declination but also the effects of changing latitude and longitude on sin H during the observation interval. Again, the total differential has to be zero because we want the combined effects to cancel each other with respect to their influence on sin H:

$$\frac{\partial(\sin H)}{\partial t} \cdot \left(d\,t + d\,Lon\right) + \frac{\partial(\sin H)}{\partial\,Lat} \cdot d\,Lat + \frac{\partial(\sin H)}{\partial\,Dec} \cdot d\,Dec \;=\; 0$$

$$-\frac{\partial(\sin H)}{\partial t} \cdot \left(d\,t + d\,Lon\right) = \frac{\partial(\sin H)}{\partial\,Lat} \cdot d\,Lat + \frac{\partial(\sin H)}{\partial\,Dec} \cdot d\,Dec$$

Differentiating sin H (altitude formula) with respect to Lat, we get:

$$\frac{\partial(\sin H)}{\partial\,Lat} \;=\; \cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t$$

Thus, the total change in t caused by the combined variations in Dec, Lat, and Lon is:

$$d\,t \;=\; \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t}\right) \cdot d\,Dec + \left(\frac{\tan Dec}{\sin t} - \frac{\tan Lat}{\tan t}\right) \cdot d\,Lat - d\,Lon$$

dLat and dLon are the infinitesimal changes in latitude and longitude caused by the vessel's movement during the observation interval. For practical purposes, we can substitute the measurable changes ΔDec, ΔLat and ΔLon for dDec, dLat and dLon (resulting in the measurable change Δt). ΔLat and ΔLon are calculated from course, C, and velocity, v, over ground and the time elapsed:

$$\Delta Lat\,['] = v\,[kn] \cdot \cos C \cdot \left(T_2\,[h] - T_1\,[h]\right)$$

$$\Delta Lon\,['] = v\,[kn] \cdot \frac{\sin C}{\cos Lat} \cdot \left(T_2\,[h] - T_1\,[h]\right)$$

$$1\,kn\,(knot) = 1\,nm/h$$

Again, the corrected second time of equal altitude is:

$$T_2^*\,[h] \;=\; = T_2\,[h] - \frac{\Delta t\,[°]}{15}$$

The longitude thus calculated refers to T_1. The longitude at T_2 is Lon+ΔLon.

The longitude error caused by changing latitude can be dramatic and requires the navigator's particular attention, even if the vessel moves at a moderate speed.

The above considerations clearly demonstrate that determining one's **exact** longitude by equal altitudes of the sun is not as simple as it seems to be at first glance, particularly on a traveling vessel. It is therefore understandable that with the development of position line navigation (including simple graphic solutions for a traveling vessel) longitude by equal altitudes became less important.

Time Sight

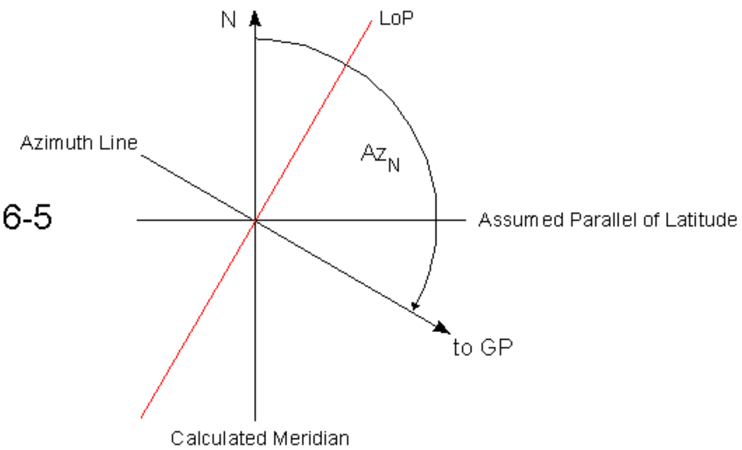
The process of deriving the longitude from a single altitude of a body (as well as the observation made for that purpose) is called **time sight**. However, this method requires knowledge of the exact latitude, e. g., a noon latitude. Solving the altitude formula (chapter 4) for the meridian angle, t, we get:

$$t = \pm \arccos \frac{\sin Ho - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

From t and GHA, we can easily calculate our longitude (see *Sumner's* method, chapter 4). In fact, *Sumner's* method is based upon multiple solutions of a time sight. During a voyage in December 1837, *Sumner* had not been able to determine the exact latitude for several days due to bad weather. One morning, when the weather finally permitted a single observation of the sun, he calculated hypothetical longitudes for three assumed latitudes. Observing that the positions thus obtained lay on a straight line which accidentally coincided with the bearing line of a terrestrial object, he realized that he had found a celestial line of position. This discovery marked the beginning of a new era of celestial navigation.

A time sight can be used to derive a line of position from a single assumed latitude. After solving the time sight, we plot the assumed parallel of latitude and the calculated meridian. Next, we calculate the azimuth of the body with respect to the position thus obtained (azimuth formula, chapter 4) and plot the azimuth line. Our line of position is the perpendicular of the azimuth line going through the calculated position (*Fig. 6-5*).

Fig. 6-5



The latter method is of historical interest only. The modern navigator will certainly prefer the intercept method (chapter 4) which can be used without any restrictions regarding meridian angle (local hour angle), latitude, and declination (see below).

A time sight is not reliable when the body is close to the meridian. Using differential calculus, we can demonstrate that the error in the meridian angle, dt, resulting from an altitude error, dH, varies in proportion with 1/sin t:

$$dt = -\frac{\cos Ho}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot dH$$

Moreover, dt varies in proportion with 1/cos Lat and 1/cos Dec. Therefore, high latitudes and declinations should be avoided as well. Of course, the same restrictions apply to *Sumner's* method.

The Meridian Angle of the Sun at Maximum Altitude

As mentioned above, the moment of maximum altitude does not exactly coincide with the upper meridian transit of the sun (or any other body) if the declination is changing. At maximum altitude, the rate of change of altitude caused by the changing declination cancels the rate of change of altitude caused by the changing meridian angle. The equation of equal altitude can be used to calculate the meridian angle of the sun at this moment. We divide each side of the equation by the infinitesimal time interval dT:

$$\frac{dt}{dT} = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t} \right) \cdot \frac{d\,Dec}{dT}$$

Measuring the rate of change of t and Dec in arcminutes per hour we get:

$$900\,'/h = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t} \right) \cdot \frac{d\,Dec\,[']}{dT\,[h]}$$

Sine t is very small, we can substitute tan t for sin t:

$$900 \approx \frac{\tan Lat - \tan Dec}{\tan t} \cdot \frac{d\,Dec\,[']}{dT\,[h]}$$

Now, we can solve the equation for tan t:

$$\tan t \approx \frac{\tan Lat - \tan Dec}{900} \cdot \frac{d\,Dec\,[']}{dT\,[h]}$$

Since a small angle (in radians) is nearly equal to its tangent, we get:

$$t\,[^\circ] \cdot \frac{\pi}{180} \approx \frac{\tan Lat - \tan Dec}{900} \cdot \frac{d\,Dec\,[']}{dT\,[h]}$$

Measuring t in arcminutes, the equation is stated as:

$$t\,['] \approx 3.82 \cdot (\tan Lat - \tan Dec) \cdot \frac{d\,Dec\,[']}{dT\,[h]}$$

dDec/dT is the rate of change of declination measured in arcminutes per hour.

The maximum altitude occurs after LAN if t is positive, and before LAN if t is negative.

For example, at the time of the spring equinox (Dec = 0, dDec/dT ≈ +1'/h) an observer being at +80° (N) latitude would observe the maximum altitude of the sun at t ≈ +21.7', i. e., 86.7 seconds after meridian transit (LAN). An observer at +45° latitude, however, would observe the maximum altitude at t ≈ +3.82', i. e., only 15.3 seconds after meridian transit.

We can use the last equation to evaluate the systematic error of a noon latitude. The latter is known to be based upon the maximum altitude, not on the meridian altitude of the sun. Following the above example, the observer at 80° latitude would observe the maximum altitude 86.7 seconds after meridian transit. During this interval, the declination of the sun would have changed from 0 to +1.445" (assuming that Dec is 0 at the time of meridian transit). Using the altitude formula (chapter 4), we get:

$$Hc = \arcsin\left(\sin 80^\circ \cdot \sin 1.445'' + \cos 80^\circ \cdot \cos 1.445'' \cdot \cos 21.7'\right) = 10^\circ\ 0'\ 0.72''$$

In contrast, the calculated altitude at meridian transit would be exactly 10°. Thus, the error of the noon latitude would be -0.72".

In the same way, we can calculate the maximum altitude of the sun observed at 45° latitude:

$$Hc = \arcsin\left(\sin 45° \cdot \sin 0.255'' + \cos 45° \cdot \cos 0.255'' \cdot \cos 3.82'\right) = 45°\ 0'\ 0.13''$$

In this case, the error of the noon latitude would be only -0.13".

The above examples show that even at the times of the equinoxes, the systematic error of a noon latitude caused by the changing declination of the sun is much smaller than other observational errors, e. g., the errors in dip or refraction. A significant error in latitude can only occur if the observer is very close to one of the poles (tan Lat!). Around the times of the solstices, the error in latitude is practically non-existent.

Chapter 7

Finding Time *and* Longitude by Lunar Observations

In navigation, time and longitude are interdependent. Determining one's longitude requires knowledge of the exact time and vice versa. This fundamental problem remained unsolved for many centuries, and old-time navigators were restricted to latitude sailing, i. e., traveling along a chosen parallel of latitude. As a result, the time of arrival could only be estimated, and many ships ran ashore at nighttime or when visibility was poor due to bad weather. Before the invention of the first reliable chronometer by John Harrison, many attempts were made to use astronomical events, e.g., solar eclipses and occultations of Jupiter moons as time marks. Although these methods were fairly accurate, many of them were impracticable at sea. Among the astronomical methods, deriving the time from **lunar distances** deserves special attention. After refined methods for the accurate prediction of the moon's apparent position became available in the 18th century, the angular distance of the moon from a chosen body, preferably one in or near the moon's path, compared with the predicted angular distance, could be utilized to determine the error of a less accurate timepiece at certain intervals in order to calibrate the instrument. This is possible since the moon's apparent position with respect to other heavenly bodies varies comparatively rapidly. The procedure was still in use in the second half of the 19th century due to the high price of precision chronometers [4].

In practice, the method of lunar distances was very complicated. It required measuring the angular distance between the moon's illuminated limb and a chosen body and the altitudes of the moon and said body at approximately the same time. This was usually done by three or four observers. Then a number of complex calculations had to be performed in order to convert the topocentric angular distance to the geocentric angular distance. These calculations included corrections for refraction (both bodies) as well as parallax and augmented semidiameter of the moon. Reportedly, it took several hours to complete this procedure called „clearing the lunar distance“.

We will not discuss the traditional method of lunar distances in detail here although it is an intellectual challenge. Instead, we will develop a less complicated way to derive the chronometer error from lunar observations using the well-known sight reduction formulas from chapter 4 and the equation of equal altitudes described in chapter 6. This method, too, uses the hourly variation in the siderial hour angle of the moon as a time standard.

GHA_{Aries} increases by 902.46 arcminutes per hour. Since siderial hour angles and declinations of fixed stars change very slowly, each stellar circle of equal altitude travels westward at about the same rate. Accordingly, a chronometer error of +1h (chronometer fast) displaces any **stellar line of position** as well as any **stellar fix** 902.46 arcminutes to the west.

The GHA of the moon increases only at a rate of 859.0+*v* arcminutes per hour. Accordingly, a chronometer error of +1h displaces any **lunar line of position** 859.0+*v* arcminutes to the west, provided the declination of the moon is constant. The small quantity *v*, the variable excess over the adopted minimum value of 859.0 arcminutes per hour, is tabulated on the daily pages of the Nautical Almanac.

As a result of the different hourly changes of GHA_{Aries} and the GHA of the moon, the siderial hour angle of the moon changes at a rate of *v*-43.46 arcminutes per hour (retrograde motion):

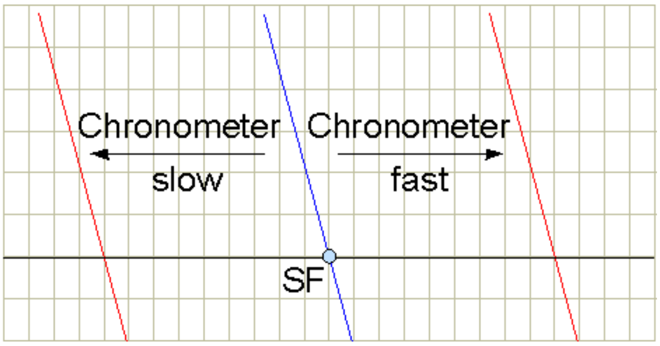
$$\Delta SHA_{Moon}[\,'\,] = (859.0 + v - 902.46)\cdot \Delta T[h] = (v - 43.46)\cdot \Delta T[h]$$

If we neglect the geometrical errors of the intercept method, a lunar line of position passes exactly through a fix derived from the altitudes of two or more stars if our chronometer shows the accurate time, provided the observations are error-free and the observer remains stationary. The lunar LoP and the stellar fix drift apart as the chronometer error increases.

Constant declination of the moon provided, the angular distance of the lunar LoP from the stellar fix, measured along the parallel of latitude going through the fix, equals the change in the siderial hour angle of the moon, ΔSHA, during the time interval ΔT, the chronometer error.

As demonstrated in *Fig. 7-1*, the lunar LoP appears eastward from the stellar fix, SF, (ΔSHA negative) if the chronometer error, ΔT, is positive (chronometer fast). The lunar LoP appears westward from stellar fix (ΔSHA positive) if the chronometer error is negative (chronometer slow).

Fig. 7-1



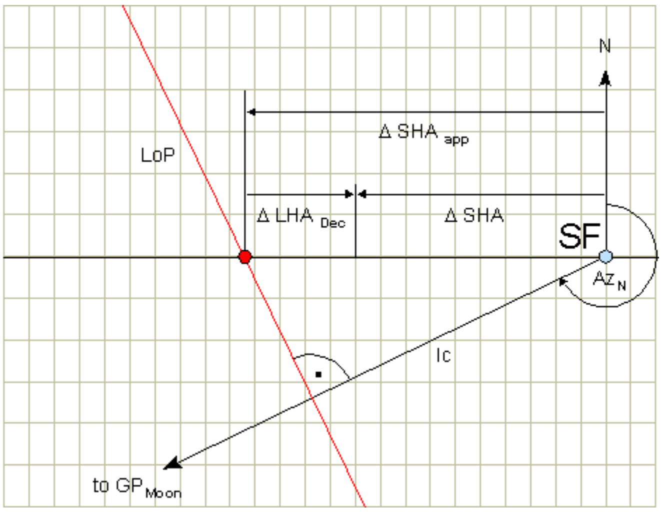
We begin with measuring the altitudes of two (or more) stars and the moon and applying the usual corrections. At each observation, we note the time as indicated by our chronometer. Measurements and altitude corrections should be made with greatest care and accuracy. Intermediate results should not be rounded but noted to the last digit. The altitude corrections for the moon should include the small correction for the oblateness of the earth (see chapter 2).

Next, we derive a fix from the star observations using the intercept method (see chapter 4). To reduce the geometrical error caused by the curvatures of the position lines, we calculate as many iteration cycles as necessary to obtain a constant position (usually 2 or 3). Of course, our stellar fix has an unknown longitude error. This is not a problem since we only evaluate differences in longitude. The obtained latitude, however, is accurate. We will need it during the further course of our calculations.

We use the fix thus obtained as our assumed position to derive the lunar line of position. Again, we apply the sight reduction formulas for the intercept method.

Fig. 7-2 shows a graphic plot of the lunar line of position. The point where the lunar LoP intersects the parallel of latitude going through the stellar fix, SF, marks ΔSHA_{app}. The latter is the westward or eastward shift of the lunar LoP with respect to SF caused by the combined effects of the chronometer error, ΔT, and the corresponding change in the declination of the moon, ΔDec.

Fig. 7-2



$$\Delta SHA_{app}\,['] = -\frac{Ic\,[nm]}{\sin Az_N \cdot \cos Lat}$$

The above formula is an approximation neglecting the curvature of the lunar LoP. The resulting error is not significant if the azimuth is in the area of 90° or 270° (±30°) and if the altitude of the moon is not too high (< ~70°). Usually, the influence of observation errors on the final result is many times greater.

ΔSHA equals ΔSHA_{app} only if the moon's declination is constant during the interval ΔT. Usually, the moon's declination changes rapidly. The quantity d tabulated on the daily pages of the Nautical Almanac is the hourly change in declination measured in arcminutes. We remember that d can be positive or negative, depending on the current trend of declination. The change in Dec during the interval ΔT is:

$$\Delta Dec\left[{}'\right] = d \cdot \Delta T\left[h\right]$$

ΔDec is equivalent to a change in the local hour angle, ΔLHA_{Dec}, if we consider equal altitudes of the moon. ΔLHA_{Dec} can be positive or negative. We can calculate ΔLHA_{Dec} with the **equation of equal altitudes** (see chapter 6):

$$\Delta LHA_{Dec} = f \cdot \Delta Dec \qquad f = \left(\frac{\tan Lat}{\sin LHA} - \frac{\tan Dec}{\tan LHA}\right)$$

LHA is the algebraic sum of the GHA of the moon and the longitude of the stellar fix.

Thus, we get:

$$\Delta LHA_{Dec}\left[{}'\right] = f \cdot d \cdot \Delta T\left[h\right]$$

ΔSHA is the algebraic sum of ΔSHA_{app} and ΔLHA_{Dec}:

$$\Delta SHA = \Delta SHA_{app} + \Delta LHA_{Dec}$$

Combining the above formulas, we have:

$$\left(v - 43.46\right) \cdot \Delta T\left[h\right] = f \cdot d \cdot \Delta T\left[h\right] - \frac{Ic\left[nm\right]}{\sin Az_N \cdot \cos Lat}$$

Solving the equation for ΔT (in seconds of time), we get the chronometer error:

$$\Delta T\left[s\right] = 3600 \cdot \frac{Ic\left[nm\right]}{\left(43.46 - v + f \cdot d\right) \cdot \sin Az_N \cdot \cos Lat}$$

According to the hourly variation of GHA_{Aries}, our improved longitude is:

$$Lon_{improved} = Lon + C \qquad C\left[{}'\right] = 0.25068 \cdot \Delta T\left[s\right]$$

Lon is the raw longitude obtained by our observations of stars.

One should have no illusions about the accuracy of longitudes obtained by lunar distances and related methods since small observation errors result in large errors in time and longitude. This is due to the fact that siderial hour angle and declination of the moon change slowly compared with the rate of change of GHA which is the basis for usual sight reduction procedures. Longitude errors of 1° were considered as normal in the days of lunar distances when nothing better was available.

The above method works best if the azimuth of the moon is 90° or 270°. In practice, an azimuth of 90°±30° or 270°±30° is acceptable. The optimum altitude of the moon is a trade-off between refraction errors (low altitude) and the curvature of the LoP (high altitudes). Therefore, medium altitudes (20°-60°) are preferred. The accuracy of the method decreases with increasing latitude. Therefore, it should not be used in polar regions. In any case, the observer has to be stationary. If the Observer's position changes during the observations, intolerable errors will result. The overall error can be reduced by multiple observations. It is therefore recommended to make a series of observations (see chapter 16).

Compared with the traditional method of lunar distances, the above procedure has not only the advantage that the required formulas are much simpler but also that it can be managed by one person since it is not necessary to make the observations at the same time. On land, relatively accurate results can be obtained when using a theodolite to measure the altitudes. In this case, the residual error in longitude usually does not exceed a few arcminutes.

Chapter 8

Rise, Set, Twilight

General Visibility

For the planning of observations, it is useful to know the times during which a certain body is above the horizon as well as the times of sunrise, sunset, and twilight.

A body can be constantly above the horizon, constantly below the horizon, or above the horizon during a part of the day, depending on the observer's latitude and the declination of the body.

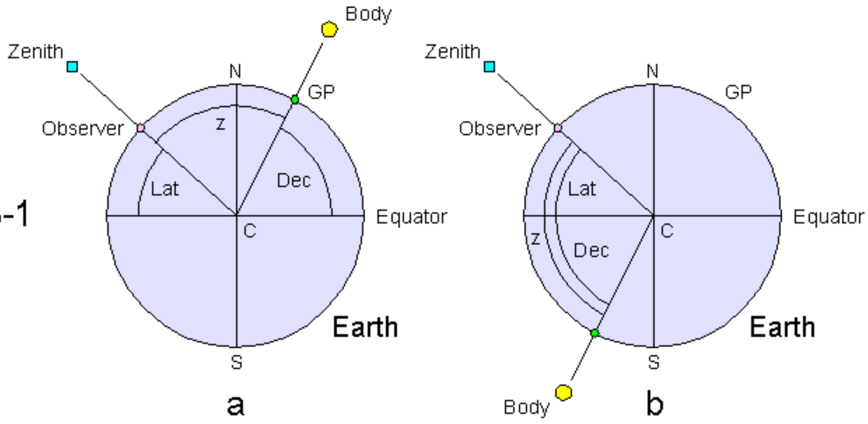
A body is **circumpolar** (always above the celestial horizon) if the zenith distance is smaller than 90° at the moment of **lower** meridian passage, i. e., when the body is on the lower branch of the local meridian (*Fig 8-1a*). This is the case under the following conditions:

$$Lat \cdot Dec > 0 \ \text{ AND } \left| Lat + Dec \right| > 90°$$

A body is continually below the celestial horizon if the zenith distance is greater than 90° at the instant of **upper** meridian passage (*Fig 8-1b*). The corresponding rule is:

$$Lat \cdot Dec < 0 \ \text{ AND } \left| Lat - Dec \right| > 90°$$

Fig. 8-1



A celestial body being on the same hemisphere as the observer is either sometimes above the horizon or circumpolar. A body being on the opposite hemisphere is either sometimes above the horizon or permanently invisible, but never circumpolar.

The sun provides a good example of how the visibility of a body is affected by latitude and declination. At the time of the summer solstice (Dec = +23.5°), the sun is circumpolar to an observer being north of the **arctic circle** (Lat > +66.5°). At the same time, the sun remains below the horizon all day if the observer is south of the **antarctic circle** (Lat < –66.5°). At the times of the equinoxes (Dec = 0°), sun sun is circumpolar only at the poles. At the time of the winter solstice (Dec = –23.5°), the sun is circumpolar south of the antarctic circle and invisible north of the arctic circle. If the observer is between the arctic and the antarctic circle, the sun is visible during a part of the day all year round.

Rise and Set

The events of rise and set can be used to determine latitude, longitude, or time. One should not expect very accurate results, however, since the atmospheric refraction may be erratic if the body is on or near the horizon.

The geometric rise or set of a body occurs when the center of the body passes through the celestial horizon (H = 0°).

Due to the influence of atmospheric refraction, all bodies except the moon appear above the visible and sensible horizon at this instant. The moon is not visible at the moment of her geometric rise or set since the depressing effect of the horizontal parallax (~1°) is greater than the elevating effect of atmospheric refraction.

The approximate apparent altitudes (referring to the sensible horizon) at the moment of the astronomical rise or set are:

Sun (lower limb):	15'
Stars:	29'
Planets:	29' – HP

When measuring these altitudes with reference to the sea horizon, we have to add the dip of horizon (chapter 2) to the above values. For example, the altitude of the lower limb of the rising or setting sun is approx. 20' if the height of eye is 8m.

We begin with the well-known altitude formula (see chapter 4), substituting the meridian angle, t, for the local hour angle (see chapter 3). We can do this since the trigonometric functions of t are equal to those of the corresponding LHA.

$$\sin H \;=\; 0 \;=\; \sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos\; t$$

$$\cos\; t \;=\; -\frac{\sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

Solving the equation for t, we get :

$$t \;=\; \arccos\; \left(-\tan Lat \cdot \tan Dec\right)$$

The equation has no solution if the argument of the inverse cosine is smaller than –1 or greater than 1. In the first case, the body is **circumpolar**, in the latter case, the body remains continuously below the horizon. Otherwise, the arccos function returns values in the range from 0°through 180°.

Due to the ambiguity of the arccos function, the equation has two solutions, one for rise and one for set. For the calculations below, we have to observe the following rules:

If the body is **rising** (body eastward from the observer), t is treated as a **negative** quantity.

If the body is **setting** (body westward from the observer), t is treated as a **positive** quantity.

If we know our latitude and the time of rise or set, we can calculate our longitude:

$$Lon \;=\; \pm t - GHA$$

GHA is the Greenwich hour angle of the body at the moment of rise or set. The sign of t has to be observed carefully (see above). If the resulting longitude is smaller than –180°, we add 360°.

Knowing our position, we can calculate the times of sunrise and sunset:

$$GMT_{Surise/set} \;=\; 12 \;\pm\; \frac{t\left[\circ\right]}{15} - \frac{Lon\left[\circ\right]}{15} - EoT$$

The times of sunrise and sunset obtained with the above formula are not quite accurate since Dec and EoT are variable. Since we do not know the exact time of rise or set at the beginning, we have to use estimated values for Dec and EoT initially. The time of rise or set can be improved by iteration (repeating the calculations with Dec and EoT at the calculated time of rise or set). Further, the times thus calculated are influenced by the irregularities of atmospheric refraction near the horizon. Therefore, a time error of ±2 minutes is not unusual.

Accordingly, we can calculate our longitude from the time of sunrise or sunset if we know our latitude:

$$Lon[°] = \pm t + 15 \cdot \left(12 - GMT_{Sunrise/set} - EoT\right)$$

Again, this is not a very precise method, and an error of several arcminutes in longitude is not unlikely.

Knowing our longitude, we are able to determine our approximate latitude from the time of sunrise or sunset:

$$t[°] = Lon[°] - 15 \cdot \left(12 - GMT_{Sunrise/set} - EoT\right)$$

$$Lat = \arctan\left(-\frac{\cos t}{\tan Dec}\right)$$

In navigation, rise and set are defined as the moments when the upper limb of a body crosses the visible horizon. These events can be observed without a sextant. Now, we have to take into account the effects of refraction, horizontal parallax, dip, and semidiameter. These quantities determine the altitude (Ho) of a body with respect to the celestial horizon at the instant of the visible rise or set.

$$t = \arccos\frac{\sin Ho - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

$$Ho = HP - SD - R_H - Dip$$

According to the Nautical Almanac, the refraction for a body being on the sensible horizon, R_H, is approximately (!) 34'.

When observing the upper limb of the sun, we get:

$$Ho = 0.15' - 16' - 34' - Dip \approx -50' - Dip$$

Ho is negative. If we refer to the upper limb of the sun and the sensible horizon (Dip=0), the meridian angle at the time of sunrise or sunset is:

$$t = \arccos\frac{-0.0145 - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

Azimuth and Amplitude

The azimuth angle of a rising or setting body is calculated with the azimuth formula (see chapter 4):

$$Az = \arccos\frac{\sin Dec - \sin H \cdot \sin Lat}{\cos H \cdot \cos Lat}$$

With H=0, we get:

$$Az = \arccos \frac{\sin Dec}{\cos Lat}$$

Az is +90° (rise) and –90° (set) if the declination of the body is zero, regardless of the observer's latitude. Accordingly, the sun rises in the east and sets in the west at the times of the equinoxes (geometric rise and set).

With H_{center}= –50' (upper limb of the sun on the sensible horizon), we have:

$$Az = \arccos \frac{\sin Dec + 0.0145 \cdot \sin Lat}{0.9999 \cdot \cos Lat}$$

The true azimuth of the rising or setting body is:

$$Az_N = \begin{cases} Az & \text{if} \quad \sin t \leq 0 \\ 360° - Az & \text{if} \quad \sin t > 0 \end{cases}$$

The azimuth of a body at the moment of rise or set can be used to find the magnetic declination at the observer's position (compare with chapter 13).

The horizontal angular distance of the rising/setting body from the east/west point on the horizon is called **amplitude** and can be calculated from the azimuth. An amplitude of E45°N, for instance, means that the body rises 45° north of the east point on the horizon.

Twilight

At sea, twilight is important for the observation of stars and planets since it is the only time when these bodies **and** the horizon are visible. By definition, there are three kinds of twilight. The altitude, H, refers to the center of the sun and the celestial horizon and marks the beginning (morning) and the end (evening) of the respective twilight.

Civil twilight: H = –6°

Nautical twilight: H = –12°

Astronomical twilight: H = –18°

In general, an altitude of the sun between –3° and –9° is recommended for astronomical observations at sea (best visibility of brighter stars and sea horizon). However, exceptions to this rule are possible, depending on the actual weather conditions.

The meridian angle for the sun at –6° altitude (center) is:

$$t = \arccos \frac{-0.10453 - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

Using this formula, we can find the approximate time for our observations (in analogy to sunrise and sunset).

As mentioned above, the simultaneous observation of stars **and** the horizon is possible during a limited time interval only.

To calculate the length of this interval, ΔT, we use the altitude formula and differentiate sin H with respect to the meridian angle, t:

$$\frac{d\left(\sin H\right)}{d\,t} = -\cos Lat \cdot \cos Dec \cdot \sin t$$

$$d\left(\sin H\right) = -\cos Lat \cdot \cos Dec \cdot \sin t \cdot d\,t$$

Substituting cosH·dH for d(sinH) and solving for dt, we get the change in the meridian angle, dt, as a function of a change in altitude, dH:

$$d\,t = -\frac{\cos H}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot d\,H$$

With H = –6° and dH = 6° (H = –3°...–9°), we get:

$$\Delta\,t\left[\circ\right] \approx -\frac{5.97}{\cos Lat \cdot \cos Dec \cdot \sin t}$$

Converting the change in the meridian angle to a time span (measured in minutes) and ignoring the sign, the equation is stated as:

$$\Delta\,T\left[m\right] \approx \frac{24}{\cos Lat \cdot \cos Dec \cdot \sin t}$$

The shortest possible time interval for our observations (Lat = 0, Dec = 0, t = 96°) lasts approx. 24 minutes. As the observer moves northward or southward from the equator, cos Lat and sin t decrease (t>90°). Accordingly, the duration of twilight increases. When t is 0° or 180°, ΔT is infinite.

This is in accordance with the well-known fact that twilight is shortest in equatorial regions and longest in polar regions.

We would obtain the same result when calculating t for H = –3° and H = –9°, respectively:

$$\Delta\,T\left[m\right] = 4 \cdot \left(t_{-9^\circ}\left[\circ\right] - t_{-3^\circ}\left[\circ\right]\right)$$

The Nautical Almanac provides tabulated values for the times of sunrise, sunset, civil twilight and nautical twilight for latitudes between –60° and +72° (referring to an observer being at the Greenwich meridian). In addition, times of moonrise and moonset are given.

Chapter 9

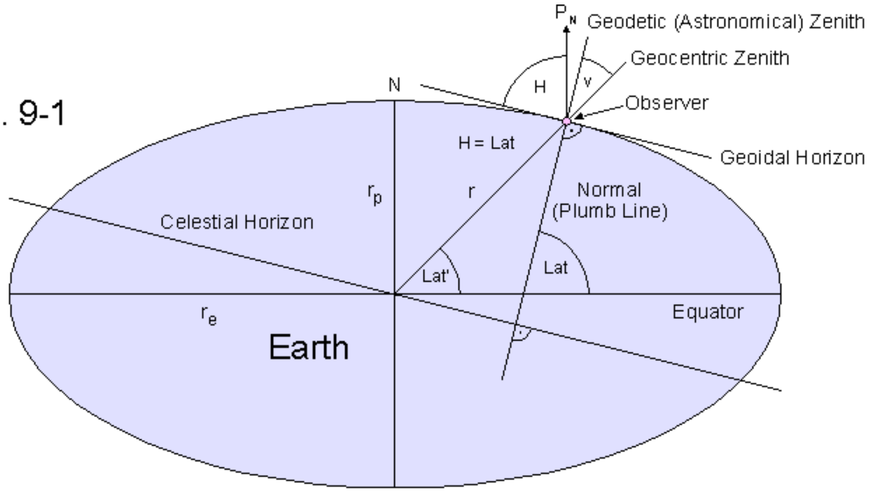
Geodetic Aspects of Celestial Navigation

The Ellipsoid

Celestial navigation is based upon the assumption that the earth is a sphere and, consequently, on the laws of spherical trigonometry. In reality, the shape of the earth is rather irregular and approximates an **oblate spheroid (ellipsoid)** resulting from two forces, **gravitation** and **centrifugal force**, acting on the viscous body of the earth. While gravitation alone would force the earth to assume the shape of a sphere, the state of lowest potential energy, the centrifugal force caused by the earth's rotation contracts the earth along the polar axis (axis of rotation) and stretches it along the plane of the equator. The local vector sum of both forces is called **gravity**.

A number of reference ellipsoids are in use to describe the shape of the earth, for example the **World Geodetic System (WGS)** ellipsoid of 1984. **The following considerations refer to the ellipsoid model of the earth which is sufficient for most navigational purposes**. *Fig.9-1* shows a meridional section of the ellipsoid.

Fig. 9-1



Earth data (WGS 84 ellipsoid) :

Equatorial radius	r_e	6378137.0 m
Polar radius	r_p	6356752.3142 m
Flattening	$(r_e - r_p) / r_e$	1/298.25722

Due to the flattening of the earth, we have to distinguish between **geographic** and **geocentric** latitude which would be the same if the earth were a sphere. The geographic (geodetic) latitude of a given position, Lat, is the angle formed by the local normal (perpendicular) to the adopted ellipsoid and the plane of the equator. The geocentric latitude, Lat', is the angle formed by the local radius vector and the plane of the equator. Geographic and geocentric latitude are interrelated as follows:

$$\tan Lat' = \frac{r_p^2}{r_e^2} \cdot \tan Lat$$

Geographic and geocentric latitude are equal at the poles and on the equator. At all other places, the geocentric latitude, Lat', is smaller than the geographic latitude, Lat. As with the spherical earth model, geographic and geocentric longitude are equal. **Maps are always based upon geographic (geodetic) coordinates.**

In the following, we will discuss the effects of the oblateness (flattening) of the earth on celestial navigation. Altitudes (or zenith distances) measured by the navigator always refer to the local plumb line which aligns itself with gravity and points to the **astronomical zenith**.

Even the visible sea horizon correlates with the astronomical zenith since the water surface is perpendicular to the local plumb line. Under the assumption of a homogeneous mass distribution throughout the ellipsoid, the plumb line coincides with the local normal to the ellipsoid which points to the **geodetic zenith**. Thus, astronomical and geodetic zenith are identical.

The **geocentric zenith** is defined as the point where the extended local radius vector of the earth intersects the celestial sphere. The angular distance of the astronomical (geodetic) zenith from the **geocentric zenith** is called **angle of the vertical**, **v**. The angle of the vertical is a function of the geographic latitude. The following formula was proposed by *Smart* [9]:

$$v\left[''\right] \approx 692.666 \cdot \sin\left(2 \cdot Lat\right) - 1.163 \cdot \sin\left(4 \cdot Lat\right) + 0.026 \cdot \sin\left(6 \cdot Lat\right)$$

The coefficients of the formula have been adapted to the proportions of the WGS 84 ellipsoid.

The angle of the vertical at a given position equals the difference between geographic and geocentric latitude (*Fig. 9-1*):

$$v = Lat - Lat'$$

The maximum value of v, occurring at 45° geographic latitude, is approx. 11.5'. Thus, the geocentric latitude of an observer being at 45°geographic latitude is only 44° 48.5'. This difference is not negligible. Therefore, the navigator has to know if the coordinates of a fix obtained by astronomical observations are geographic or geocentric. Altitudes are measured with respect to the sea horizon or an artificial horizon. Both correlate with the local plumb line which points to the astronomical zenith. Thus, the latter is the only reference available to the navigator. As demonstrated in *Fig. 9-1*, the altitude of the celestial north pole, P_N, (corrected altitude of Polaris) with respect to the geoidal horizon equals the geographic, not the geocentric latitude. A noon latitude, being the sum or difference of the (geocentric) declination and the zenith distance with respect to the astronomical zenith would give the same result.

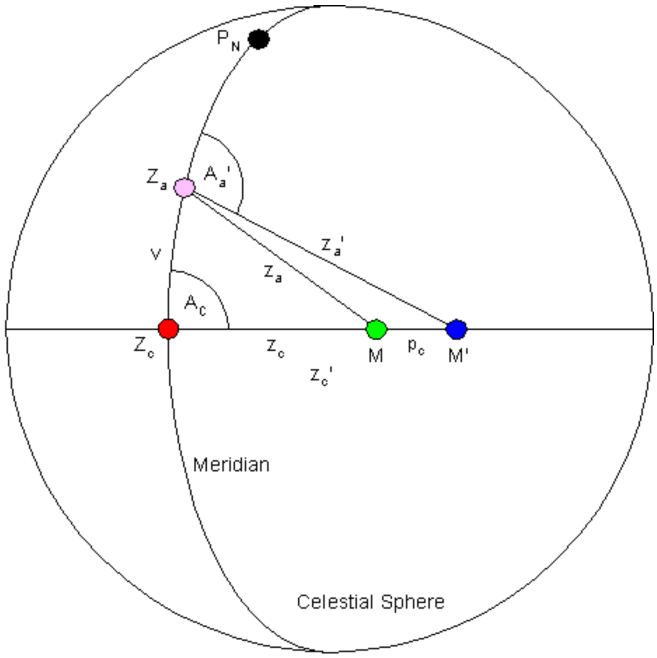
Latitudes obtained by celestial observations are geographic latitudes since the navigator measures altitudes with respect to the local astronomical zenith (directly or indirectly).

It is further important to know if the oblateness of the earth causes significant errors due to the fact that calculations of celestial navigation are based on a spherical earth model. According to the above values for polar radius and equatorial radius of the earth, the great circle distance of 1' is 1.849 km at the poles and 1.855 km at the equator. This small difference does not produce a significant error when plotting lines of position. It is therefore sufficient to use the adopted mean value (1 nautical mile = 1.852 km). However, when calculating the great circle distance (see chapter 11) of two locations thousands of nautical miles apart, the error caused by the oblateness of the earth can increase to several nautical miles. If extraordinary precision is required, the formulas for geodetic distance given in [2] should be used.

The Parallax of the Moon

Due to the oblateness of the earth, the distance between geoidal and celestial horizon is not constant but can assume any value between r_p and r_e, depending on the observer's latitude. This has a measurable effect on the parallax of the moon since tabulated values for HP refer to the equatorial radius, r_e. The parallax of the moon is further affected by the displacement of the plumb line from the earth's center. A correction formula compensating both effects is given in chapter 2. The asymmetry of the plumb line with respect to the earth's center even causes a small (negligible) parallax in azimuth unless the moon is on the local meridian. In the following, we will calculate the effects of the oblateness of the earth on the parallax of the moon with the exact formulas of spherical astronomy [9]. For practical navigation, the simplified correction formulas given in chapter 2 are accurate enough. *Fig.* 9-2 shows a projection of the astronomical zenith, Z_a, the geocentric zenith, Z_c, and the geographic position of the moon, M, on the **celestial sphere**, an imaginary hollow sphere of infinite diameter with the earth at its center.

Fig. 9-2



The geocentric zenith, Z_c, is the point where a straight line from the earth's center through the observer's position intersects the celestial sphere. The astronomical zenith, Z_a, is the point at which the plumb line going through the observer's position intersects the celestial sphere. Z_a and Z_c are on the same meridian. M is the projected geocentric position of the moon defined by Greenwich hour angle and declination. Unfortunately, the position of a body defined by GHA and Dec is commonly called geographic position (see chapter 3) although GHA and Dec are geocentric coordinates. M' is the point where a straight line from the observer through the moon's center intersects the celestial sphere. Z_c, M, and M' are on a great circle. The zenith distance measured by the observer is z_a' because the astronomical zenith is the available reference. The quantity we want to know is z_a, the astronomical zenith distance corrected for parallax in altitude. This is the angular distance of the moon from the astronomical zenith, measured by a fictitious observer at the earth's center.

The known quantities are v, A_a', and z_a'. In contrast to the astronomer, the navigator is usually not able to measure A_a' precisely. For navigational purposes, the calculated azimuth (see chapter 4) may be substituted for A_a'.

We have three spherical triangles, Z_aZ_cM', Z_aZ_cM, and Z_aMM'. First, we calculate z_c' from z_a', v, and A_a' using the **law of cosines for sides** (see chapter 10):

$$\cos z_c' = \cos z_a' \cdot \cos v + \sin z_a' \cdot \sin v \cdot \cos\left(180° - A_a'\right)$$

$$z_c' = \arccos\left(\cos z_a' \cdot \cos v - \sin z_a' \cdot \sin v \cdot \cos A_a'\right)$$

To obtain z_c, we first have to calculate the relative length ($r_e = 1$) of the radius vector, r, and the geocentric parallax, p_c:

$$\frac{r}{r_e} = \sqrt{\frac{1-\left(2e^2 - e^4\right) \cdot \sin^2 Lat}{1 - e^2 \cdot \sin^2 Lat}} \qquad e^2 = 1 - \frac{r_p^2}{r_e^2}$$

$$p_c = \arcsin\left(\frac{r}{r_e} \cdot \sin HP \cdot \sin z_c'\right)$$

HP is the equatorial horizontal parallax.

The geocentric zenith distance corrected for parallax is:

$$z_c = z_c' - p_c$$

Using the cosine formula again, we calculate A_c, the azimuth angle of the moon with respect to the geocentric zenith:

$$A_c = \arccos \frac{\cos z_a' - \cos z_c' \cdot \cos v}{\sin z_c' \cdot \sin v}$$

The astronomical zenith distance corrected for parallax is:

$$z_a = \arccos\left(\cos z_c \cdot \cos v + \sin z_c \cdot \sin v \cdot \cos A_c\right)$$

Thus, the **parallax in altitude** (astronomical) is:

$$PA = z_a' - z_a$$

The small angle between M and M', measured at Z_a, is the **parallax in azimuth**, p_{az}:

$$p_{az} = \arccos \frac{\cos p_c - \cos z_a \cdot \cos z_a'}{\sin z_a \cdot \sin z_a'}$$

Comparing the exact formulas with the simplified parallax formulas given in chapter 2 (including the correction for the oblateness of the earth), we will find very small differences in the observed altitude (small fractions of an arcsecond). These small differences are not measurable with the instruments of celestial navigation and can be neglected.

The parallax in azimuth does not exist when the moon is on the local meridian and is greatest when the moon ist east or west of the observer. It is further greatest at medium latitudes (45°) and non-existant when the observer is at one of the poles or on the equator (v = 0). Even under the most unfavourable conditions, the parallax in azimuth is only a fraction of an arcminute and therefore insignificant to celestial navigation.

Other celestial bodies do not require a correction for the oblateness of the earth since their parallaxes are very small compared with the parallax of the moon.

The Geoid

The earth is not **exactly** an oblate ellipsoid. The shape of the earth is more accurately described by the **geoid**, an equipotential surface of the earth's field of gravity. This surface has elevations and depressions caused by geographic features and a non-uniform mass distribution (materials of different density). Elevations occur at local accumulations of matter (mountains, ore deposits), depressions at local deficiencies of matter (valleys, lakes, caverns). The elevation or depression of each point of the geoid with respect to the ellipsoid is found by gravity measurement. At the flank of an elevation or depression of the geoid, the plumb line (the normal to the geoid) does not coincide with the normal to the ellipsoid, and the astronomical zenith differs from the geodetic zenith. Thus, the astronomical latitude or longitude of a position (obtained through astronomical observations) may slightly differ from the geographic (geodetic) latitude or longitude. These differences are usually smaller than one arcminute, but greater local differences have been reported, for example, in coastal waters with adjacent high mountains.

The errors caused by the irregular shape of the geoid are usually not relevant to celestial navigation at sea but are important to surveying and map-making where a higher degree of accuracy is required.

Chapter 10

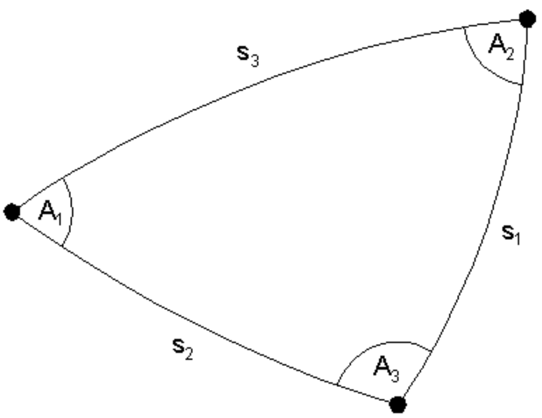
Spherical Trigonometry

The earth is usually regarded as a sphere in celestial navigation although an oblate spheroid would be a better approximation. Otherwise, navigational calculations would become too difficult for practical use. The position error introduced by the spherical earth model is usually very small and stays within the "statistical noise" caused by other omnipresent errors like, e.g., abnormal refraction, rounding errors, etc. Although it is possible to perform navigational calculations solely with the aid of tables (H.O. 229, H.O. 211, etc.) and with little mathematics, the principles of celestial navigation can not be understood without knowing the elements of spherical trigonometry.

The Oblique Spherical Triangle

A spherical triangle is – like any triangle – characterized by three sides and three angles. Unlike a plane triangle, a spherical triangle is part of the surface of a sphere, and the sides are not straight lines but arcs of great circles (*Fig. 10-1*).

Fig. 10-1



A **great circle** is a circle on the surface of a sphere whose plane passes through the center of the sphere (see chapter 3).

Any side of a spherical triangle can be regarded as an angle - the angular distance between the adjacent vertices, measured at the center of the sphere. The interrelations between angles and sides of a spherical triangle are described by the **law of sines**, the **law of cosines for sides**, the **law of cosines for angles**, and **Napier's analogies** (apart from other formulas).

Law of sines:

$$\frac{\sin A_1}{\sin s_1} = \frac{\sin A_2}{\sin s_2} = \frac{\sin A_3}{\sin s_3}$$

Law of cosines for sides:

$$\cos s_1 = \cos s_2 \cdot \cos s_3 + \sin s_2 \cdot \sin s_3 \cdot \cos A_1$$
$$\cos s_2 = \cos s_1 \cdot \cos s_3 + \sin s_1 \cdot \sin s_3 \cdot \cos A_2$$
$$\cos s_3 = \cos s_1 \cdot \cos s_2 + \sin s_1 \cdot \sin s_2 \cdot \cos A_3$$

Law of cosines for angles:

$$\cos A_1 = -\cos A_2 \cdot \cos A_3 + \sin A_2 \cdot \sin A_3 \cdot \cos s_1$$
$$\cos A_2 = -\cos A_1 \cdot \cos A_3 + \sin A_1 \cdot \sin A_3 \cdot \cos s_2$$
$$\cos A_3 = -\cos A_1 \cdot \cos A_2 + \sin A_1 \cdot \sin A_2 \cdot \cos s_3$$

Napier's analogies:

$$\frac{\tan \frac{A_1 + A_2}{2}}{\cot \frac{A_3}{2}} = \frac{\cos \frac{s_1 - s_2}{2}}{\cos \frac{s_1 + s_2}{2}} \qquad \frac{\tan \frac{A_1 - A_2}{2}}{\cot \frac{A_3}{2}} = \frac{\sin \frac{s_1 - s_2}{2}}{\sin \frac{s_1 + s_2}{2}}$$

$$\frac{\tan \frac{s_1 + s_2}{2}}{\cot \frac{s_3}{2}} = \frac{\cos \frac{A_1 - A_2}{2}}{\cos \frac{A_1 + A_2}{2}} \qquad \frac{\tan \frac{s_1 - s_2}{2}}{\cot \frac{s_3}{2}} = \frac{\cos \frac{A_1 - A_2}{2}}{\cos \frac{A_1 + A_2}{2}}$$

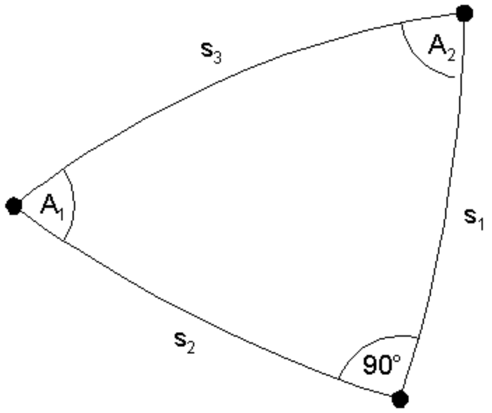
These formulas and others derived thereof enable any quantity (angle or side) of a spherical triangle to be calculated if three other quantities are known.

Particularly the law of cosines for sides is of interest to the navigator.

The Right Spherical Triangle

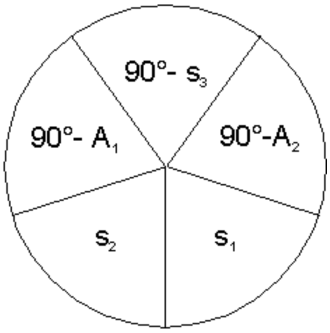
Solving a spherical triangle gets simpler if it contains a right angle (*Fig. 10-2*). Using ***Napier's*** **Rules of Circular Parts**, any quantity can be calculated if only two other quantities (apart from the right angle) are known.

Fig. 10-2



We arrange the sides forming the right angle (s_1, s_2) and the **complements** of the remaining angles (A_1, A_2) and opposite side (s_3) in the form of a pie chart consisting of five sectors, called "parts" (in the same order as they occur in the triangle). The right angle itself is omitted (*Fig. 10-3*):

Fig. 10-3



According to *Napier's* rules, the sine of any part of the diagram equals the product of the tangents of the adjacent parts and the product of the cosines of the opposite parts:

$$
\begin{aligned}
\sin s_1 &= \tan s_2 \cdot \tan\left(90° - A_2\right) = \cos\left(90° - A_1\right) \cdot \cos\left(90° - s_3\right) \\
\sin s_2 &= \tan\left(90° - A_1\right) \cdot \tan s_1 = \cos\left(90° - s_3\right) \cdot \cos\left(90° - A_2\right) \\
\sin\left(90° - A_1\right) &= \tan\left(90° - s_3\right) \cdot \tan s_2 = \cos\left(90° - A_2\right) \cdot \cos s_1 \\
\sin\left(90° - s_3\right) &= \tan\left(90° - A_2\right) \cdot \tan\left(90° - A_1\right) = \cos s_1 \cdot \cos s_2 \\
\sin\left(90° - A_2\right) &= \tan s_1 \cdot \tan\left(90° - s_3\right) = \cos s_2 \cdot \cos\left(90° - A_1\right)
\end{aligned}
$$

In a simpler form, these equations are stated as:

$$
\begin{aligned}
\sin s_1 &= \tan s_2 \cdot \cot A_2 = \sin A_1 \cdot \sin s_3 \\
\sin s_2 &= \cot A_1 \cdot \tan s_1 = \sin s_3 \cdot \sin A_2 \\
\cos A_1 &= \cot s_3 \cdot \tan s_2 = \sin A_2 \cdot \cos s_1 \\
\cos s_3 &= \cot A_2 \cdot \cot A_1 = \cos s_1 \cdot \cos s_2 \\
\cos A_2 &= \tan s_1 \cdot \cot s_3 = \cos s_2 \cdot \sin A_1
\end{aligned}
$$

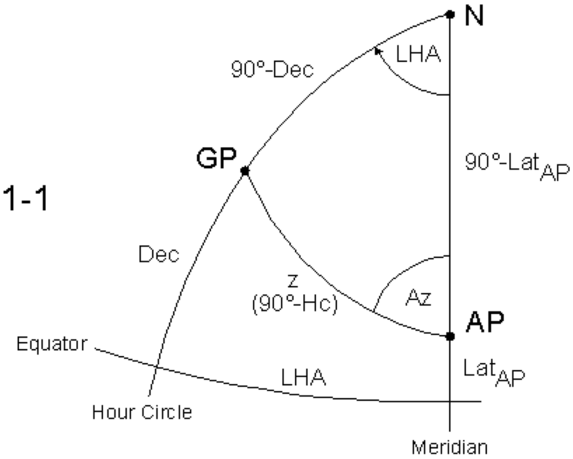
Sight reduction tables (chapter 11) are based upon the formulas of the right spherical triangle.

Chapter 11

The Navigational Triangle

The **navigational triangle**, the basis of all common sight reduction procedures, is the (usually oblique) spherical triangle on the earth's surface formed by the north pole, N, the observer's assumed position, AP, and the geographic position of the celestial object, GP, at the time of observation (*Fig. 11-1*).

Fig. 11-1



When using the intercept method, the latitude of the assumed position, Lat_{AP}, the declination of the observed celestial body, Dec, and the local hour angle, LHA, or the meridian angle, t, (calculated from the longitude of AP and the GHA of the object), are initially known to the observer.

The first step is calculating the side z of the navigational triangle by using the law of cosines for sides:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos LHA$$

Since cos (90°-x) equals sin x and vice versa, the equation can be written in a simpler form:

$$\cos z = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

The side z is not only the **great circle distance** between AP and GP but also the **zenith distance** of the celestial object and the radius of the **circle of equal altitude** (see chapter 1).

Substituting the altitude H for z, we get:

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

Solving the equation for H leads to the altitude formula known from chapter 4:

$$H = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA\right)$$

The azimuth angle of the observed body is also calculated by means of the law of cosines for sides:

$$\cos\left(90° - Dec\right) = \cos\left(90° - Lat_{AP}\right) \cdot \cos z + \sin\left(90° - Lat_{AP}\right) \cdot \sin z \cdot \cos Az$$

$$\sin Dec = \sin Lat_{AP} \cdot \cos z + \cos Lat_{AP} \cdot \sin z \cdot \cos Az$$

Using the altitude, which has been calculated already, instead of the zenith distance, results in the following equation:

$$\sin Dec = \sin Lat_{AP} \cdot \sin H + \cos Lat_{AP} \cdot \cos H \cdot \cos Az$$

Solving the equation for Az finally yields the azimuth formula from chapter 4:

$$Az = \arccos \frac{\sin Dec - \sin Lat_{AP} \cdot \sin H}{\cos Lat_{AP} \cdot \cos H}$$

The resulting azimuth angle is always in the range of 0°... 180° and therefore not necessarily identical with the true azimuth, Az_N (0°... 360° clockwise from true north) commonly used in navigation. In all cases where sin LHA is negative (GP east of AP) , Az equals Az_N. Otherwise (sin LHA positive, GP westward from AP , as shown in *Fig. 11-1*), Az_N is obtained by subtracting Az from 360°.

When using Sumner's method, z (or H), Dec, and Lat_{AP} (the assumed latitude) are initially known, and LHA (or t) is the quantity to be calculated.

Again, the law of cosines for sides can be applied:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos LHA$$

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

$$\cos LHA = \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

$$LHA = \arccos \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

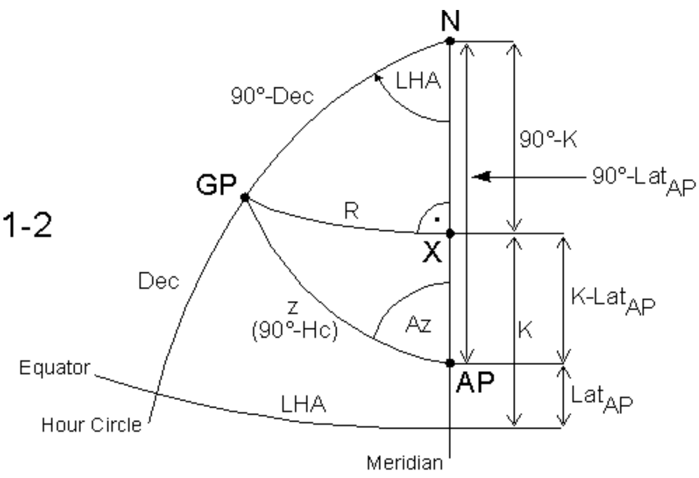
The obtained LHA (or t) is then processed as described in chapter 4.

When observing a celestial body at the time of meridian passage, the local hour angle is zero, and the navigational triangle becomes infinitesimally narrow. In this special case, the formulas of spherical trigonometry are not needed, and the sides of the spherical triangle can be calculated by simple addition or subtraction.

The Divided Navigational Triangle

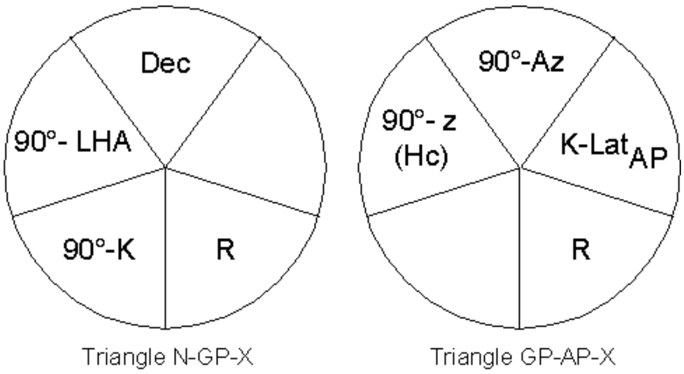
An alternative, very elegant method of solving the navigational triangle begins with dividing it in two right spherical triangles by constructing a great circle passing through GP and intersecting the meridian going through AP at a right angle (*Fig. 11-2*):

Fig. 11-2



The first right triangle is formed by GP, N, and X, the second one by GP, X, and AP. The **auxiliary parts** R and K are intermediate quantities used to calculate z (or Hc) and Az. Both triangles are solved using **Napier's Rules of Circular Parts** (see chapter 9). *Fig. 11-3* illustrates the corresponding circular diagrams:

Fig. 11-3



According to Napier's rules, Hc and Az are calculated by means of the following formulas:

$$\sin R = \sin LHA \cdot \cos Dec \quad \Rightarrow \quad R = \arcsin\left(\sin LHA \cdot \cos Dec\right)$$

$$\sin Dec = \cos R \cdot \sin K \quad \Rightarrow \quad \sin K = \frac{\sin Dec}{\cos R} \quad \Rightarrow \quad K = \arcsin \frac{\sin Dec}{\cos R}$$

$$\sin Hc = \cos R \cdot \cos\left(K - Lat_{AP}\right) \quad \Rightarrow \quad Hc = \arcsin\left[\cos R \cdot \cos\left(K - Lat_{AP}\right)\right]$$

$$\sin R = \cos Hc \cdot \sin Az \quad \Rightarrow \quad \sin Az = \frac{\sin R}{\cos Hc} \quad \Rightarrow \quad Az = \arcsin \frac{\sin R}{\cos Hc}$$

To obtain the true azimuth, Az_N (0°... 360°), the following rules have to be applied:

If the sine of LHA is negative, enter the above formulas with the explementary angle, 360°-LHA, instead of LHA.

$$Az_N = \begin{cases} Az & \text{if} \quad \sin LHA < 0 \text{ AND } Dec > Lat_{AP} \\ 180° - Az & \text{if} \quad \sin LHA < 0 \text{ AND } Dec < Lat_{AP} \\ 180° + Az & \text{if} \quad \sin LHA > 0 \text{ AND } Dec < Lat_{AP} \\ 360° - Az & \text{if} \quad \sin LHA > 0 \text{ AND } Dec > Lat_{AP} \end{cases}$$

The divided navigational triangle is of considerable importance since it forms the theoretical background for a number of **sight reduction tables**, e.g., the Ageton Table.

Using the secant and cosecant function (sec x = 1/cos x, csc x = 1/sin x) and substituting the meridian angle, t, for LHA, the equations for the divided navigational triangle are stated as:

$$\csc R = \csc t \cdot \sec Dec$$

$$\csc K = \frac{\csc Dec}{\sec R}$$

$$\csc Hc = \sec R \cdot \sec(K - Lat)$$

$$\csc Az = \frac{\csc R}{\sec Hc}$$

In logarithmic form, these equations are stated as:

$$\log \csc R = \log \csc t + \log \sec Dec$$

$$\log \csc K = \log \csc Dec - \log \sec R$$

$$\log \csc Hc = \log \sec R + \log \sec (K - Lat)$$

$$\log \csc Az = \log \csc R - \log \sec Hc$$

Having the logarithms of the secants and cosecants of angles available in the form of a suitable table, we can solve a sight by a sequence of simple additions and subtractions (beside converting the angles to their corresponding log secants and log cosecants and vice versa). Apart from the table itself, the only tools required are a sheet of paper and a pencil.

The *Ageton* Table (H.O. 211), first published in 1931, is based upon the above formulas and provides a very efficient arrangement of angles and their log secants and log cosecants on 36 pages as well as step by step instructions for use.

Sight reduction tables were developed many years before the invention of electronic calculators in order to simplify calculations necessary to reduce a sight. Still today, sight reduction tables are preferred by people who do not want to deal with the formulas of spherical trigonometry. Moreover, they provide a valuable backup method if electronic devices fail.

A modified version of the *Ageton* Table is available at: http://www.umland.onlinehome.de/page3.htm

Chapter 12

Other Navigational Formulas

The following formulas - although not part of celestial navigation - are of vital interest because they enable the navigator to calculate course and distance from initial positon A to final position B as well as to calculate the final position B from initial position A, course, and distance.

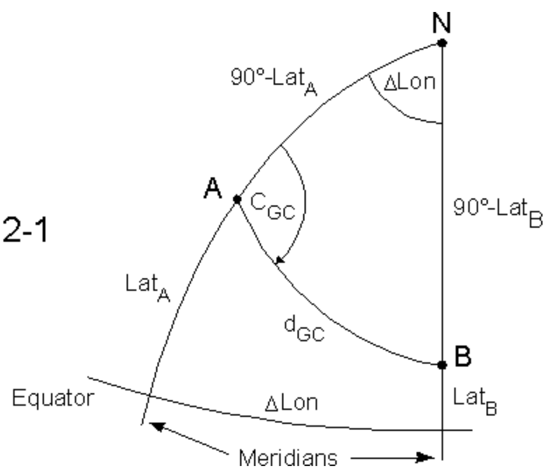
Calculation of Course and Distance

If the coordinates of the initial position A, Lat_A and Lon_A, and the coordinates of the final position B (destination), Lat_B and Lon_B, are known, the navigator has the choice of either traveling along the great circle going through A and B (shortest route) or traveling along the rhumb line going through A and B (slightly longer but easier to navigate).

Great Circle

Great circle distance d_{GC} and course C_{GC} are derived from the navigational triangle (chapter 11) by substituting A for GP, B for AP, d_{GC} for z, and ΔLon (= Lon_B-Lon_A) for LHA (*Fig. 12-1*):

Fig. 12-1



$$d_{GC} = \arccos\left[\sin Lat_A \cdot \sin Lat_B + \cos Lat_A \cdot \cos Lat_B \cdot \cos\left(Lon_B - Lon_A\right)\right]$$

(Northern latitude and eastern longitude are positive, southern latitude and western longitude negative.)

$$C_{GC} = \arccos \frac{\sin Lat_B - \sin Lat_A \cdot \cos d_{GC}}{\cos Lat_A \cdot \sin d_{GC}}$$

C_{GC} has to be converted to the explementary angle, 360°-C_{GC}, if sin (Lon_B-Lon_A) is negative, in order to obtain the true course (0°... 360° clockwise from true north).

C_{GC} is only the initial course and has to be adjusted either continuously or at appropriate intervals because with changing position the angle between the great circle and each local meridian also changes (unless the great circle is the equator or a meridian itself).

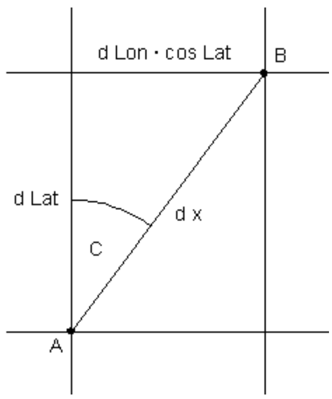
d_{GC} has the dimension of an angle. To convert it to a distance, we multiply d_{GC} by 40031.6/360 (yields the distance in km) or by 60 (yields the distance in nm).

Rhumb Line

A **rhumb line (loxodrome)** is a line on the surface of the earth intersecting all meridians at a constant angle. A vessel steering a constant compass course travels along a rhumb line, provided there is no drift and the magnetic variation remains constant. Rhumb line course C_{RL} and distance d_{RL} are calculated as follows:

First, we imagine traveling the infinitesimally small distance dx from the point of departure, A, to the point of arrival, B. Our course is C (*Fig. 12-2*):

Fig. 12-2



The path of travel, dx, can be considered as composed of a north-south component, dLat, and a west-east component, dLon · cos Lat. The factor cos Lat is the relative circumference of the respective parallel of latitude (equator = 1):

$$\tan C = \frac{d\,Lon \cdot \cos Lat}{d\,Lat}$$

$$\frac{d\,Lat}{\cos Lat} = \frac{1}{\tan C} \cdot d\,Lon$$

If we increase the distance between A (Lat_A, Lon_A) and B (Lat_B, Lon_B), we have to integrate:

$$\int_{Lat\ A}^{Lat\ B} \frac{d\ Lat}{\cos\ Lat} = \frac{1}{\tan\ C} \cdot \int_{Lon\ A}^{Lon\ B} d\ Lon$$

$$\ln\left[\tan\left(\frac{Lat_B}{2} + \frac{\pi}{4}\right)\right] - \ln\left[\tan\left(\frac{Lat_A}{2} + \frac{\pi}{4}\right)\right] = \frac{Lon_B - Lon_A}{\tan C}$$

$$\tan C = \frac{Lon_B - Lon_A}{\ln \frac{\tan\left(\frac{Lat_B}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{Lat_A}{2} + \frac{\pi}{4}\right)}}$$

Solving for C and measuring angles in degrees, we get:

$$C_{RL} = \arctan \frac{\pi \cdot \left(Lon_B - Lon_A\right)}{180° \cdot \ln \frac{\tan\left(\frac{Lat_B[°]}{2} + 45°\right)}{\tan\left(\frac{Lat_A[°]}{2} + 45°\right)}}$$

(Lon_B-Lon_A) has to be in the range from -180° to +180°. If it is outside this range, add or subtract 360° before entering the rhumb line course formula.

The arctan function returns values between -90° and +90°. To obtain the true course, $C_{RL,N}$, we apply the following rules:

$$C_{RL,N} = \begin{cases} C_{RL} & \text{if} \quad Lat_B > Lat_A \quad \text{AND} \quad Lon_B > Lon_A \\ 180° - C_{RL} & \text{if} \quad Lat_B < Lat_A \quad \text{AND} \quad Lon_B > Lon_A \\ 180° + C_{RL} & \text{if} \quad Lat_B < Lat_A \quad \text{AND} \quad Lon_B < Lon_A \\ 360° - C_{RL} & \text{if} \quad Lat_B > Lat_A \quad \text{AND} \quad Lon_B < Lon_A \end{cases}$$

To find the total length of our path of travel, we calculate the infinitesimal distance dx:

$$d\,x = \frac{d\,Lat}{\cos C}$$

The total length is found through integration:

$$D = \int_0^D d\,x = \frac{1}{\cos C} \cdot \int_{Lat\,A}^{Lat\,B} d\,Lat = \frac{Lat_B - Lat_A}{\cos C}$$

Measuring D in kilometers or nautical miles, we get:

$$D_{RL}[km] = \frac{40031.6}{360} \cdot \frac{Lat_B - Lat_A}{\cos C_{RL}} \qquad D_{RL}[nm] = 60 \cdot \frac{Lat_B - Lat_A}{\cos C_{RL}}$$

If both positions have the same latitude, the distance can not be calculated using the above formulas. In this case, the following formulas apply (C_{RL} is either 90° or 270°.):

$$D_{RL}\left[km\right] = \frac{40031.6}{360} \cdot \left(Lon_B - Lon_A\right) \cdot \cos Lat \qquad D_{RL}\left[nm\right] = 60 \cdot \left(Lon_B - Lon_A\right) \cdot \cos Lat$$

Mid latitude

Since the rhumb line course formula is rather complicated, it is mostly replaced by the **mid latitude** formula in everyday navigation. This is an approximation giving good results as long as the distance between both positions is not too large and both positions are far enough from the poles.

Mid latitude course:

$$C_{ML} = \arctan\left(\cos Lat_M \cdot \frac{Lon_B - Lon_A}{Lat_B - Lat_A}\right) \qquad Lat_M = \frac{Lat_A + Lat_B}{2}$$

The true course is obtained by applying the same rules to C_{ML} as to the rhumb line course C_{RL}.

Mid latitude distance:

$$d_{ML}\left[km\right] = \frac{40031.6}{360} \cdot \frac{Lat_B - Lat_A}{\cos C_{ML}} \qquad d_{ML}\left[nm\right] = 60 \cdot \frac{Lat_B - Lat_A}{\cos C_{ML}}$$

If C_{ML} = 90° or C_{ML} = 270°, apply the following formulas:

$$d_{ML}\left[km\right] = \frac{40031.6}{360} \cdot \left(Lon_B - Lon_A\right) \cdot \cos Lat \qquad d_{ML}\left[nm\right] = 60 \cdot \left(Lon_B - Lon_A\right) \cdot \cos Lat$$

Dead Reckoning

Dead reckoning is the navigational term for calculating one's new position B (**dead reckoning position, DRP**) from the previous position A, course C, and distance d (calculated from the vessel's average speed and time elapsed). Since dead reckoning can only yield an approximate position (due to the influence of drift, etc.), the mid latitude method provides sufficient accuracy. On land, dead reckoning is more difficult than at sea since it is usually not possible to steer a constant course (apart from driving in large, entirely flat areas like, e.g., salt flats). At sea, the DRP is usually used to choose an appropriate (near-by) AP. If celestial observations are not possible and electronic navigation aids are not available, dead reckoning may be the only way of keeping track of one's position.

Calculation of new latitude:

$$Lat_B\left[\circ\right] = Lat_A\left[\circ\right] + \frac{360}{40031.6} \cdot d\left[km\right] \cdot \cos C \qquad Lat_B\left[\circ\right] = Lat_A\left[\circ\right] + \frac{d\left[nm\right]}{60} \cdot \cos C$$

Calculation of new longitude:

$$Lon_B\left[\circ\right] = Lon_A\left[\circ\right] + \frac{360}{40031.6} \cdot d\left[km\right] \cdot \frac{\sin C}{\cos Lat_M} \qquad Lon_B\left[\circ\right] = Lon_A\left[\circ\right] + \frac{d\left[nm\right]}{60} \cdot \frac{\sin C}{\cos Lat_M}$$

If the resulting longitude exceeds 180°, subtract 360°. If it exceeds -180°, add 360°.

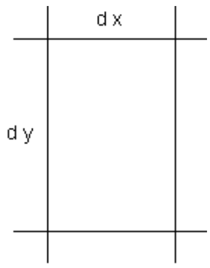
Chapter 13

Mercator Charts and Plotting Sheets

Sophisticated navigation is almost impossible without the use of a map (chart), a projection of a certain area of the earth's surface on a plane sheet of paper. There are several types of map projection, but the **Mercator projection,** named after the German cartographer Gerhard Kramer (Latin: Gerardus Mercator), is mostly used in navigation because it produces charts with an orthogonal grid which is most convenient for measuring directions and plotting lines of position. Further, rhumb lines appear as straight lines on a Mercator chart. Great circles do not, apart from meridians and the equator which are also rhumb lines.

In order to construct a Mercator chart, we have to remember how the grid printed on a globe looks. At the equator, an area of, e. g., 5 by 5 degrees looks almost like a square, but it becomes an increasingly narrow trapezoid as we move toward one of the poles. While the distance between two adjacent parallels of latitude remains constant, the distance between two meridians becomes progressively smaller as the latitude increases. An area with the infinitesimally small dimensions dLat and dLon would appear as an oblong with the dimensions dx and dy on our globe (*Fig. 13-1*):

Fig. 13-1



$$dx = c' \cdot d\,Lon \cdot \cos Lat$$

$$dy = c' \cdot d\,Lat$$

dx contains the factor cos Lat since the circumference of a parallel of latitude is in direct proportion to cos Lat. The constant c' is the scale of the globe (measured in, e. g., mm/°).

Since we require **any** rhumb line to appear as a **straight** line intersecting all meridians at a constant angle, meridians have to be equally spaced vertical lines on our chart, and an infinitesimally small oblong defined by dLat and dLon must have a constant aspect ratio, regardless of its position on the chart (dy/dx = const.).

Therefore, if we transfer the oblong defined by dLat and dLon from the globe to our chart, we get the dimensions:

$$dx = c \cdot d\,Lon$$

$$dy = c \cdot \frac{d\,Lat}{\cos Lat}$$

The new constant c is the scale of the chart. Now, dx remains constant (parallel meridians) but dy is a function of the latitude at which our small oblong is located. To obtain the smallest distance between any point at the latitude Lat_P and the equator, we integrate:

$$Y = \int_0^Y dy = c \cdot \int_0^{Lat_P} \frac{d\,Lat}{\cos\,Lat} = c \cdot \ln\tan\left(\frac{Lat_P}{2} + \frac{\pi}{4}\right)$$

Y is the distance of the respective parallel of latitude from the equator. In the above equation, angles are given in circular measure (radians). If we measure angles in degrees, the equation is stated as:

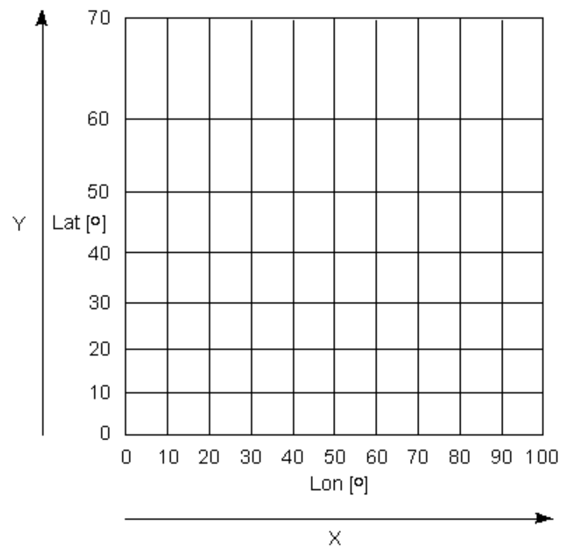
$$Y = c \cdot \ln \tan\left(\frac{Lat_P[°]}{2} + 45° \right)$$

The distance of any point from the Greenwich meridian (Lon = 0°) varies proportionally with the longitude of the point, Lon_P. X is the distance of the respective meridian from the Greenwich meridian:

$$X = \int_0^{Lon_P} dx = c \cdot Lon_P$$

Fig. 13-2 shows an example of the resulting grid. While meridians of longitude appear as equally spaced vertical lines, parallels of latitude are horizontal lines drawn farther apart as the latitude increases. Y would be infinite at 90° latitude.

Fig. 13-2



Mercator charts have the disadvantage that geometric distortions increase as the distance from the equator increases. The Mercator projection is therefore not suitable for polar regions. A circle of equal altitude, for example, would appear as a distorted ellipse at higher latitudes. Areas near the poles, e. g., Greenland, appear much greater on a Mercator map than on a globe.

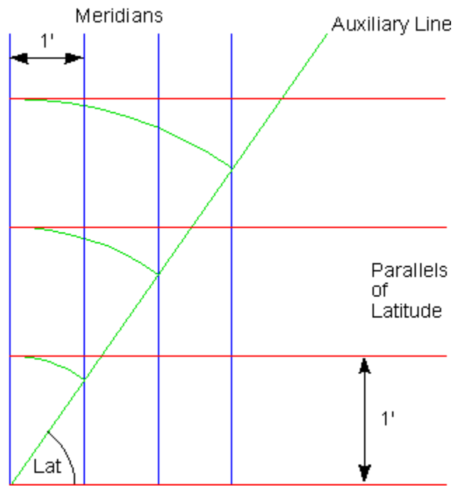
It is often said that a Mercator chart is obtained by projecting each point of the surface of a globe along lines radiating from the center of the globe to the inner surface of a hollow cylinder tangent to the globe at the equator. This is only a rough approximation. As a result of such a projection, Y would be proportional to tan Lat, and the aspect ratio of a small oblong defined by dLat and dLon would vary, depending on its position on the chart.

If we magnify a small part of a Mercator chart, e. g., an area of 30' latitude by 40' longitude, we will notice that the spacing between the parallels of latitude now seems to be almost constant. An approximated Mercator grid of such a small area can be constructed by drawing equally spaced horizontal lines, representing the parallels of latitude, and equally spaced vertical lines, representing the meridians. The spacing of the parallels of latitude, Δy, defines the scale of our chart, e. g., 5mm/nm. The spacing of the meridians, Δx, is a function of the middle latitude, Lat_M, the latitude represented by the horizontal line going through the center of our sheet of paper:

$$\Delta x = \Delta y \cdot \cos Lat_M$$

A sheet of paper with such a simplified Mercator grid is called a **small area plotting sheet** and is a very useful tool for plotting lines of position (*Fig. 13-3*).

Fig. 13-3



If a calculator or trigonometric table is not available, the meridian lines can be constructed with the following graphic method:

We take a sheet of blank paper and draw the required number of equally spaced horizontal lines (parallels). A spacing of 3 - 10 mm per nautical mile is recommended for most applications.

We draw an auxiliary line intersecting the parallels of latitude at an angle equal to the mid latitude. Then we mark the map scale, e.g., 5 mm/nm, periodically on this line, and draw the meridian lines through the points thus located (*Fig. 13-3*). Compasses can be used to transfer the map scale to the auxiliary line.

Small area plotting sheets are available at nautical book stores.

A useful program (shareware) for printing small area plotting sheets for any given latitude between 0° and 80° can be downloaded from this web site:

http://perso.easynet.fr/~philimar/graphpapeng.htm (present URL)

Chapter 14

Magnetic Declination

Since the magnetic poles of the earth do not coincide with the geographic poles and due to other irregularities of the earth's magnetic field, the needle of a magnetic compass, aligning itself with the horizontal component of the magnetic lines of force, usually does not point exactly in the direction of the geographic north pole. The angle between the direction of the compass needle and the local geographic meridian (true north) is called **magnetic declination** or, in mariner's language, **variation** (*Fig. 14-1*).

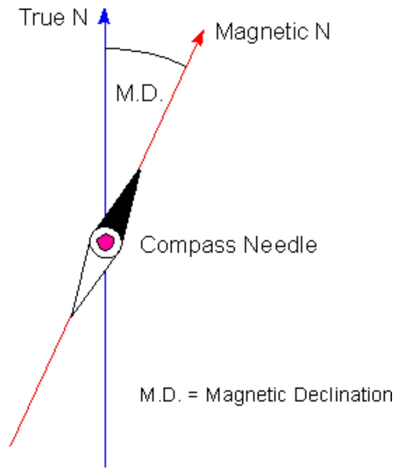


Fig. 14-1

Magnetic declination depends on the observer's geographic position and can exceed 30° or even more in some areas. The knowledge of the local magnetic declination is therefore important to avoid dangerous navigation errors. Although magnetic declination is often listed in the legend of topographic maps, the information may be outdated because magnetic declination slowly changes with time (up to several degrees per decade). In some places, magnetic declination may even differ from official statements due to local anomalies of the magnetic field caused by deposits of ferromagnetic ores, etc.

The azimuth formulas described in chapter 4 provide a powerful tool to determine the magnetic declination if the observer's position is known. A sextant is not required for the simple procedure:

1. We choose a celestial body being low in the sky or on the visible horizon, preferably sun or moon. We measure the compass bearing of the center of the body and note the time. We stay away from steel objects and DC power cables.

2. We extract GHA and Dec of the body from the N.A.

3. We calculate the LHA using our actual longitude. If the actual longitude is not known, we use the estimated longitude.

4. We calculate the azimuth, Az_N, of the body and subtract the azimuth from the compass bearing.

The difference is the magnetic declination at our position, provided the compass is error-free. Eastern declination (shown in *Fig. 14-1*) is positive, western negative. If the magnetic declination is known, the method can be used to determine the compass error.

Chapter 15

Ephemerides of the Sun

The sun is probably the most frequently observed body in celestial navigation. Greenwich hour angle and declination of the sun as well as GHA_{Aries} and EoT can be calculated using the algorithms listed below. The formulas are relatively simple and useful for navigational calculations with programmable pocket calculators (10 digits recommended).

First, the time variable, T, has to be calculated from year, month, and day. T is the number of days before or after Jan 1, 2000, 12:00:00 GMT:

$$T = 367 \cdot y - \mathrm{int}\left\{1.75 \cdot \left[y + \mathrm{int}\left(\frac{m+9}{12}\right)\right]\right\} + \mathrm{int}\left(275 \cdot \frac{m}{9}\right) + d + \frac{GMT}{24} - 730531.5$$

y is the number of the year (4 digits), m is the number of the month, and d the number of the day of the respective month. GMT (UT) is Greenwich mean time in decimal format (e.g., 12h 30m 45s = 12.5125). For May 17, 1999, 12:30:45 GMT, for example, T is -228.978646. The equation is valid from March 1, 1900 through February 28, 2100.

Mean anomaly of the sun*:

$$g[\circ] = 0.9856003 \cdot T - 2.472$$

Mean longitude of the sun*:

$$L_M[\circ] = 0.9856474 \cdot T - 79.53938$$

True longitude of the sun*:

$$L_T[\circ] = L_M[\circ] + 1.915 \cdot \sin g + 0.02 \cdot \sin(2 \cdot g)$$

Obliquity of the ecliptic:

$$\varepsilon[\circ] = 23.439 - 4 \cdot 10^{-7} \cdot T$$

Declination of the sun:

$$Dec[\circ] = \arcsin(\sin L_T \cdot \sin\varepsilon)$$

Right ascension of the sun (in degrees)*:

$$RA[\circ] = 2 \cdot \arctan\left(\frac{\cos\varepsilon \cdot \sin L_T}{\cos Dec + \cos L_T}\right)$$

GHA_{Aries}*:

$$GHA_{Aries}[\circ] = 0.9856474 \cdot T + 15 \cdot GMT + 100.46062$$

Greenwich hour angle of the sun*:

$$GHA[°] = GHA_{Aries} - RA[°]$$

Equation of time:

$$EoT[m] = 4 \cdot \left(L_M[°] - RA[°] \right)$$

*These quantities have to be within the range from 0° through 360°. If necessary, add or subtract 360° or multiples thereof. This can be achieved using the following algorithm which is particularly useful for programmable calculators:

$$y = 360 \cdot \left[\frac{x}{360} - \mathrm{int}\left(\frac{x}{360} \right) \right]$$

int(x) is the greatest integer smaller than x. For example, int(3.8) = 3, int(-2.2) = -3. The ***int*** function is called ***floor*** in some programming languages, e.g., JavaScript.

Accuracy

Unfortunately, no information on accuracy is given in the original literature [8]. Therefore, results have been cross-checked with *Interactive Computer Ephemeris 0.51* (accurate to approx. 0.1'). Between the years 1900 and 2049, no difference greater than ±0.5' for GHA and Dec was found with 100 dates chosen at random. In most cases, the error was less than ±0.3'. EoT was accurate to approx. ±2s. In comparison, the maximum error in GHA and Dec extracted from the Nautical Almanac is approx. ±0.25' when using the interpolation tables.

Semidiameter and Horizontal Parallax

Due to the excentricity of the earth's orbit, semidiameter and horizontal parallax of the sun change periodically during the course of a year. The SD of the sun is calculated using the following formula:

$$SD['] = 16 + 0.27 \cdot \cos \frac{30.4 \cdot (m-1) + d - 3}{1.015}$$

The argument of the cosine is stated in degrees.

The mean HP of the sun is 8.8 arcseconds. The periodic variation of HP is too small to be of practical significance.

Chapter 16

Navigational Errors

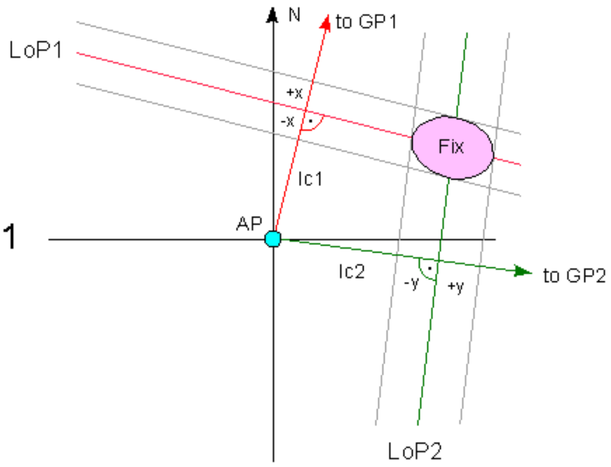
Altitude errors

Apart from systematic errors which can be corrected to a large extent (see chapter 2), observed altitudes always contain random errors caused by ,e.g., heavy seas, abnormal atmospheric refraction, and limited optical resolution of the human eye. Although a good sextant has a mechanical accuracy of ca. 0.1'- 0.3', the **standard deviation** of an altitude measured with a marine sextant is approximately 1' under fair working conditions. The standard deviation may increase to several arcminutes due to disturbing factors or if a bubble sextant or a plastic sextant is used. Altitudes measured with a theodolite are considerably more accurate (0.1'- 0.2').

Due to the influence of random errors, lines of position become indistinct and are better considered as **bands of position**.

Two intersecting bands of position define an **area of position** (ellipse of uncertainty). *Fig. 16-1* illustrates the approximate size and shape of the ellipse of uncertainty for a given pair of LoP's. The standard deviations (±x for the first altitude, ±y for the second altitude) are indicated by grey lines.

Fig. 16-1



The area of position is smallest if the angle between the bands is 90°. The most probable position is at the center of the area, provided the error distribution is symmetrical. Since LoP's are perpendicular to their corresponding azimuth lines, objects should be chosen whose azimuths differ by approx. 90° for best accuracy. An angle between 30° and 150°, however, is tolerable in most cases.

When observing more than two bodies, the azimuths should have a roughly symmetrical distribution (**bearing spread**).

We divide 360° by the number of observed bodies to obtain the optimum horizontal angle between each two adjacent bodies (3 bodies: 120°, 4 bodies: 90°, 5 bodies: 72°, 6 bodies: 60°, etc.).

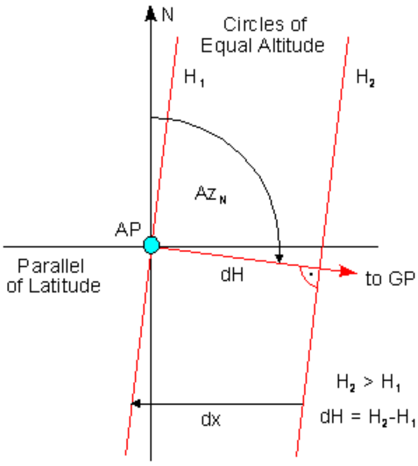
A symmetrical bearing spread not only improves geometry but also compensates for systematic errors like, e.g., index error.

Moreover, there is an optimum range of altitudes the navigator should choose to obtain reliable results. Low altitudes increase the influence of abnormal refraction (random error), whereas high altitudes, corresponding to circles of equal altitude with small diameters, increase geometric errors due to the curvature of LoP's. The generally recommended range to be used is 20° - 70°, but exceptions are possible.

Time errors

The time error is as important as the altitude error since the navigator usually presets the instrument to a chosen altitude and records the time when the image of the body coincides with the reference line visible in the telescope. The accuracy of time measurement is usually in the range between a fraction of a second and several seconds, depending on the rate of change of altitude and other factors. Time error and altitude error are closely interrelated and can be converted to each other, as shown below (*Fig. 16-2*):

Fig. 16-2



The GP of any celestial body travels westward with an angular velocity of approx. 0.25' per second. This is the rate of change of the LHA of the observed body caused by the earth's rotation. The same applies to each circle of equal altitude surrounding GP (tangents shown in *Fig. 6-2*). The distance between two concentric circles of equal altitude (with the altitudes H_1 and H_2) passing through AP in the time interval dt, measured along the parallel of latitude going through AP is:

$$dx\,[nm] = 0.25 \cdot \cos Lat_{AP} \cdot dt\,[s]$$

dx is also the east-west displacement of a LoP caused by the time error dt. The letter d indicates a small (infinitesimal) change of a quantity (see mathematical literature). cos Lat_{AP} is the ratio of the circumference of the parallel of latitude going through AP to the circumference of the equator (Lat = 0).

The corresponding difference in altitude (the radial distance between both circles of equal altitude) is:

$$dH\,['] = \sin Az_N \cdot dx\,[nm]$$

Thus, the rate of change of altitude is:

$$\frac{dH\,[']}{dt\,[s]} = 0.25 \cdot \sin Az_N \cdot \cos Lat_{AP}$$

dH/dt is greatest when the observer is on the equator and decreases to zero as the observer approaches one of the poles. Further, dH/dt is greatest if GP is exactly east of AP (dH/dt positive) or exactly west of AP (dH/dt negative). dH/dt is zero if the azimuth is 0° or 180°. This corresponds to the fact that the altitude of the observed body passes through a minimum or maximum at the instant of meridian transit (dH/dt = 0).

The maximum or minimum of altitude occurs exactly at meridian transit only if the declination of a body is constant. Otherwise, the highest or lowest altitude is observed shortly before or after meridian transit (see chapter 6). The phenomenon is particularly obvious when observing the moon whose declination changes rapidly.

A **chronometer error** is a systematic time error. It influences each line of position in such a way that only the longitude of a fix is affected whereas the latitude remains unchanged, provided the declination does not change significantly (moon!). A chronometer being 1 s fast, for example, displaces a fix 0.25' to the west, a chronometer being 1 s slow displaces the fix by the same amount to the east. If we know our position, we can calculate the chronometer error from the difference between our true longitude and the longitude found by our observations. If we do not know our longitude, the approximate chronometer error can be found by lunar observations (chapter 7).

Ambiguity

Poor geometry may not only decrease accuracy but may even result in an entirely wrong fix. As the observed horizontal angle (difference in azimuth) between two objects approaches 180°, the distance between the points of intersection of the corresponding circles of equal altitude becomes very small (at exactly 180°, both circles are tangent to each other). Circles of equal altitude with small diameters resulting from high altitudes also contribute to a short distance. A small distance between both points of intersection, however, increases the risk of ambiguity (*Fig. 16-3*).

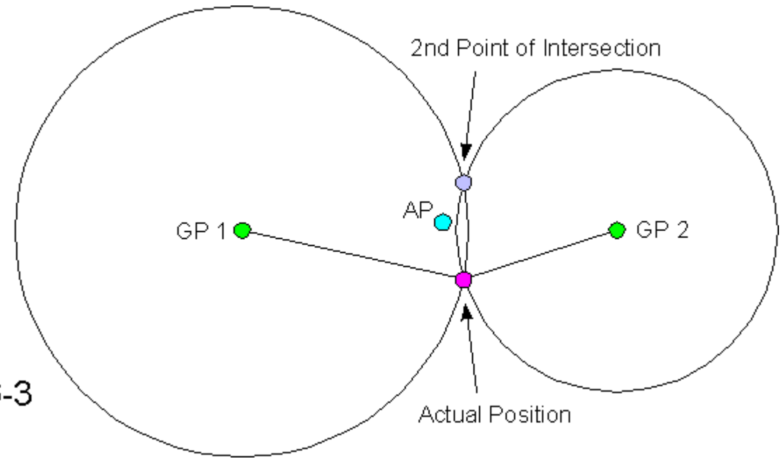


Fig. 16-3

In cases where – due to a horizontal angle near 180° and/or very high altitudes – the distance between both points of intersection is too small, we can not be sure that the assumed position is always close enough to the actual position.

If AP is close to the actual position, the fix obtained by plotting the LoP's (tangents) will be almost identical with the actual position. The accuracy of the fix decreases as the distance of AP from the actual position becomes greater. The distance between fix and actual position increases dramatically as AP approaches the line going through GP1 and GP2 (draw the azimuth lines and tangents mentally). In the worst case, a position error of several hundred or even thousand nm may result !

If AP is exactly on the line going through GP1 and GP2, i.e., equidistant from the actual position and the second point of intersection, the horizontal angle between GP1 and GP2, as viewed from AP, will be 180°. In this case, both LoP's are parallel to each other, and no fix can be found.

As AP approaches the second point of intersection, a fix more or less close to the latter is obtained. Since the actual position and the second point of intersection are symmetrical with respect to the line going through GP1 and GP2, the intercept method can not detect which of both theoretically possible positions is the right one.

Iterative application of the intercept method can only improve the fix if the initial AP is closer to the actual position than to the second point of intersection. Otherwise, an "improved" wrong position will be obtained.

Each navigational scenario should be evaluated critically before deciding if a fix is reliable or not. The distance from AP to the observer's actual position has to be considerably smaller than the distance between actual position and second point of intersection. This is usually the case if the above recommendations regarding altitude, horizontal angle, and distance between AP and actual position are observed.

A simple method to improve the reliability of a fix

Each altitude measured with a sextant, theodolite, or any other device contains systematic and random errors which influence the final result (fix). Systematic errors are more or less eliminated by careful calibration of the instrument used. The influence of random errors decreases if the number of observations is sufficiently large, provided **the error distribution** is symmetrical. Under practical conditions, the number of observations is limited, and the error distribution is more or less unsymmetrical, particularly if an **outlier**, a measurement with an abnormally large error, is present. Therefore, the average result may differ significantly from the true value. When plotting more than two lines of position, the experienced navigator may be able to identify outliers by the shape of the error polygon and remove the associated LoP's. However, the method of least squares, producing an average value, does not recognize outliers and may yield an inaccurate result.

The following simple method takes advantage of the fact that the **median** of a number of measurements is much less influenced by outliers than the **mean** value:

1. We choose a celestial body and measure a series of altitudes. We calculate azimuth and intercept for each observation of said body. The number of measurements in the series has to be odd (3, 5, 7...). The reliability of the method increases with the number of observations.

2. We sort the calculated **intercepts** by magnitude and choose the **median** (the central value in the array of intercepts thus obtained) and its associated azimuth. We discard all other observations of the series.

3. We repeat the above procedure with at least one additional body (or with the same body after its azimuth has become sufficiently different).

4. We plot the lines of position using the azimuth and intercept selected from each series, or use the selected data to calculate the fix with the method of least squares (chapter 4).

The method has been checked with excellent results on land. At sea, where the observer's position usually changes continually, the method has to be modified by advancing AP according to the path of travel between the observations of each series.

Appendix

Literature :

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